Compositeness of near-threshold exotic hadrons in systems with Coulomb and short-range interactions

T. Kinugawa and T. Hyodo, arXiv:2604.17813 [hep-ph].

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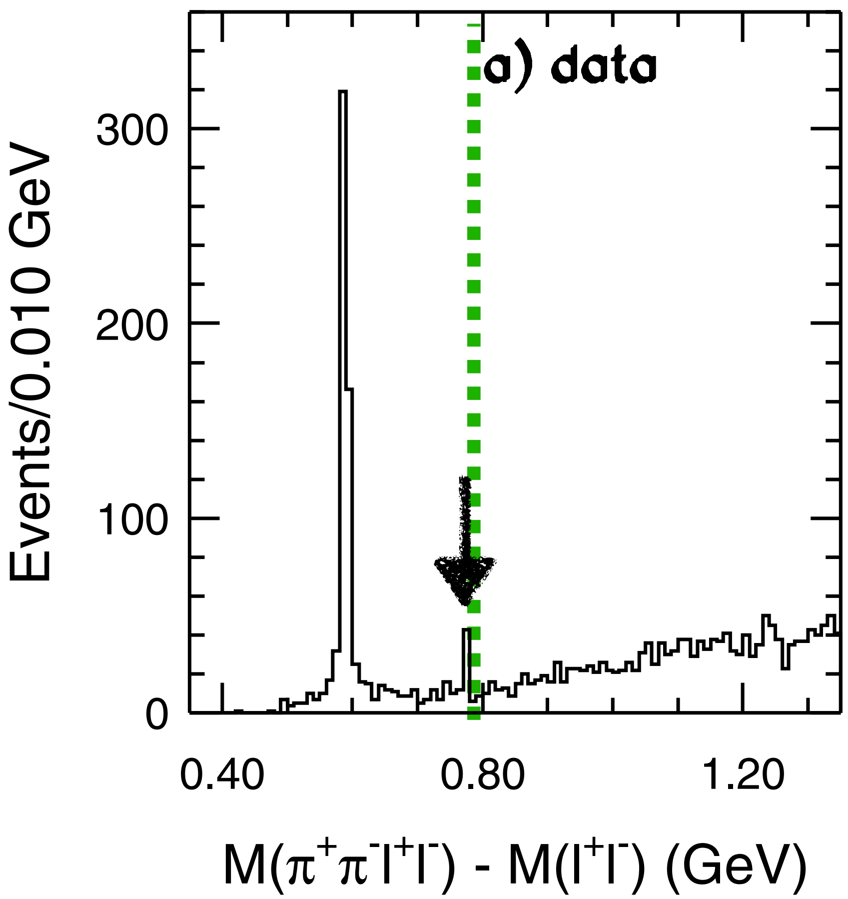
Jun 26th, Meson2026 @Kraków

Near-threshold exotic hadrons

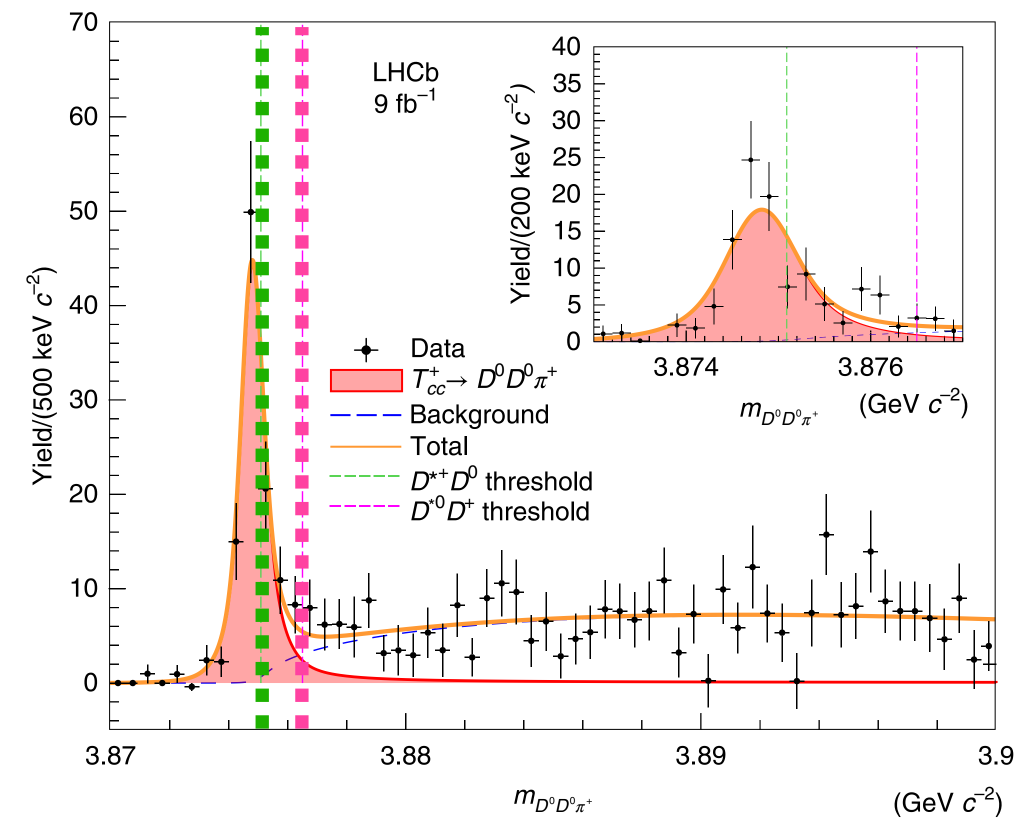
- exotic hadrons are tend to be observed near threshold

$X(3872) \to \pi^+\pi^- J/\psi$



S. K. Choi *et al*. (Belle), Phys. Rev. Lett. **91**, 262001 (2003).

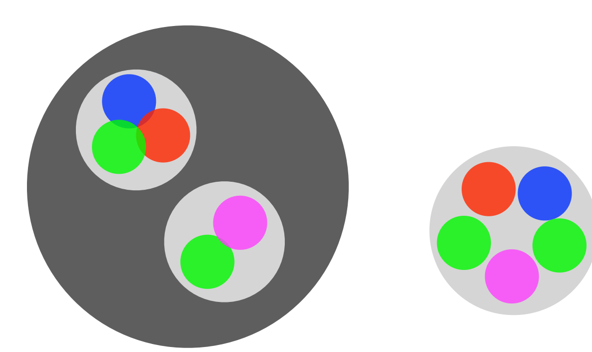
$T_{cc}(3875)^+ \to D^0 D^0 \pi^+ \ (cc\bar{u}\bar{d})$



LHCb Collaboration, Nature Phys. **18** (2022) no.7, 751-754;
LHCb Collaboration, Nat. Commun. **13** 3351 (2022).

→ internal structure of near-threshold states?

hadronic molecule, multiquark, etc…?

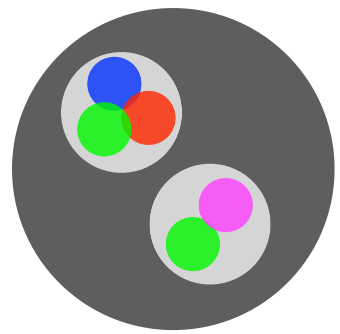


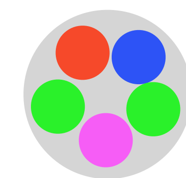
F-K. Guo, C. Hanhart, Ulf-G. Meißner, Q. Wang, Q. Zhao, and B-S. Zou, RevModPhys.90.015004 (2018).

Compositeness

S. Weinberg, Phys. Rev. 137, 672–678 (1965);
T. Hyodo, Int. J. Mod. Phys. A 28, 1330045 (2013);
U. van Kolck, Symmetry 14, 1884 (2022);
T. Kinugawa, T. Hyodo, Eur. Phys. J. A 61, 154 (2025).

◉ Definition





$$|\text{state}\rangle = \sqrt{X}\,|\text{composite}\rangle + \sqrt{Z}\,|\text{others}\rangle$$

compositeness (X) **elementarity** (Z)

- quantitative measure

$$0 \leq X \leq 1$$
$$X + Z = 1$$

→

$X > 0.5 \Leftrightarrow$ **composite dominant**

$X < 0.5 \Leftrightarrow$ **non-composite dominant**

- applicable to various states

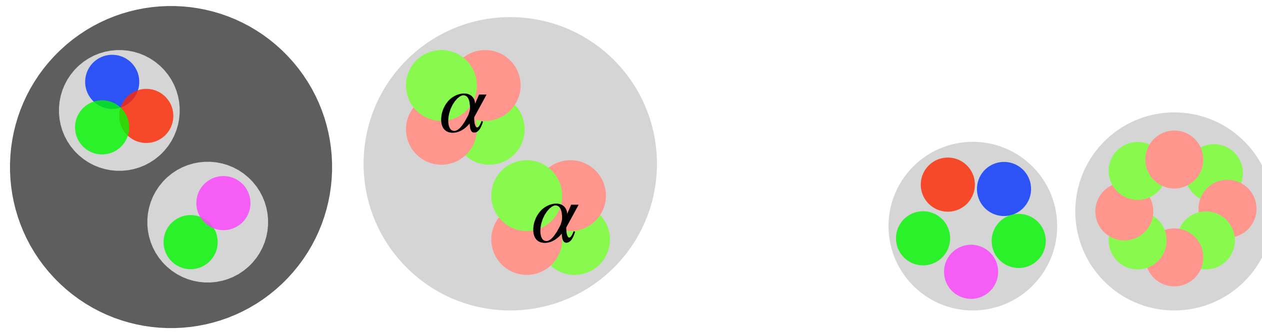
$f_0(980),\ a_0(980)$

V. Baru, J. Haidenbauer, C. Hanhart, Y. Kalashnikova, A.E. Kudryavtsev, Phys. Lett. B 586, 53–61 (2004);
Y. Kamiya and T. Hyodo, PTEP 2017; Phys. Rev. C 93, 035203 (2016) etc.

Compositeness

S. Weinberg, Phys. Rev. 137, 672–678 (1965);
T. Hyodo, Int. J. Mod. Phys. A 28, 1330045 (2013);
U. van Kolck, Symmetry 14, 1884 (2022);
T. Kinugawa, T. Hyodo, Eur. Phys. J. A 61, 154 (2025).

◉ Definition



$$|\text{state}\rangle = \sqrt{X}\,|\text{composite}\rangle + \sqrt{Z}\,|\text{others}\rangle$$

compositeness **elementarity**

- **quantitative measure**

$$0 \le X \le 1 \qquad X + Z = 1$$

→ $X > 0.5 \Leftrightarrow$ **composite dominant**

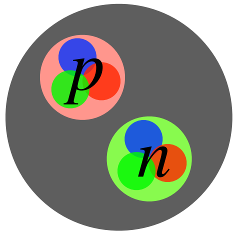
→ $X < 0.5 \Leftrightarrow$ **non-composite dominant**

- applicable to various states

$f_0(980)$, $a_0(980)$ — V. Baru, J. Haidenbauer, C. Hanhart, Y. Kalashnikova, A.E. Kudryavtsev, Phys. Lett. B 586, 53–61 (2004); Y. Kamiya and T. Hyodo, PTEP 2017; Phys. Rev. C 93, 035203 (2016) etc.

nuclei, atomic systems — E. Braaten, H. W. Hammer, and M. Kusunoki, (2003), cond-mat/0301489; T. Kinugawa, T. Hyodo, Phys. Rev. C 106, 015205 (2022) etc.

Near-threshold states



◉ near-threshold s-wave states with **short range interaction**

- near-threshold bound states

  composite dominant

  purely composite ($X \to 1$) in $B \to 0$ limit

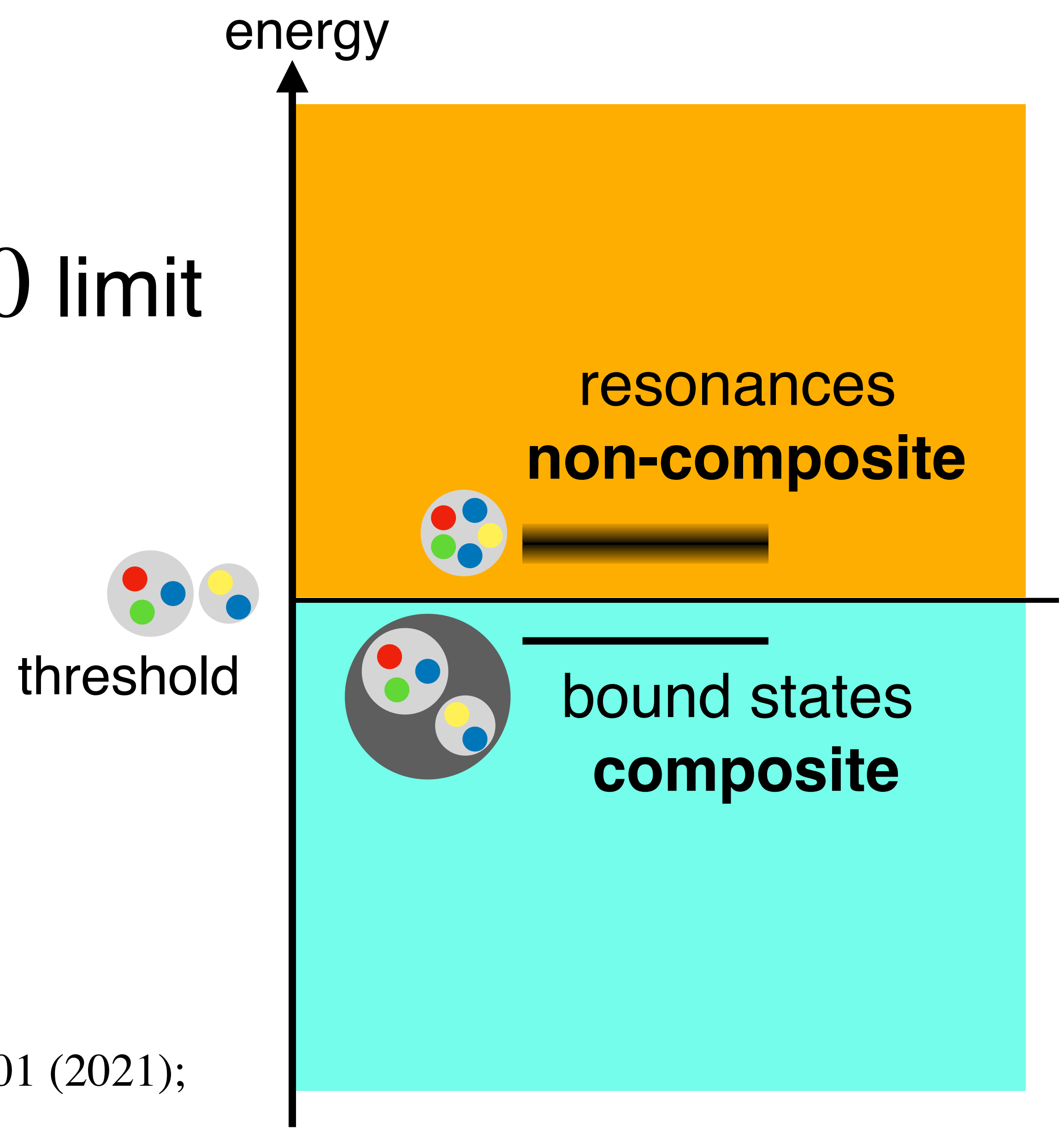
  ∵ low-energy universality
  with large scattering length

  T. Hyodo, Phys. Rev. C **90**, 055208 (2014);
  C. Hanhart, J. R. Pelaez, and G. Rios, Phys. Lett. B **739**, 375 (2014);
  T. Kinugawa and T. Hyodo Phys. Rev. C 109 , 045205 (2024).

- near-threshold resonances

  non-composite dominant

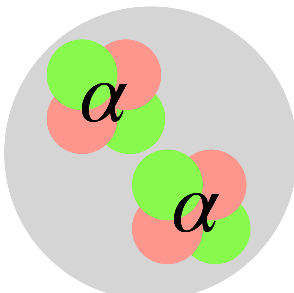
  I. Matuschek, V. Baru, F.K. Guo, C. Hanhart, Eur. Phys. J. A 57(3), 101 (2021);
  T. Kinugawa and T. Hyodo, arXiv:2403.12635 [hep-ph].

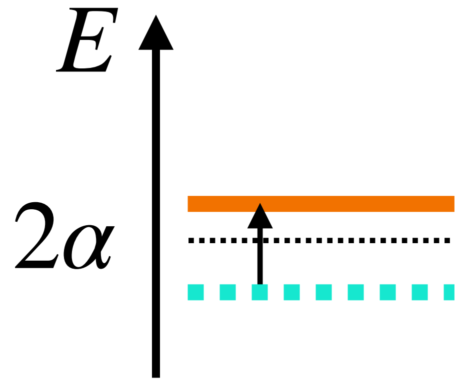


→ structure of near-threshold states depends on whether state is located below or above threshold

Coulomb + short range systems

- ^{8}Be nuclei ($2\alpha^{++}$) J. Hiura, and R. Tamagaki, Prog. Theor. Phys. Suppl. No. 52, 25 (1972); R. Higa, H.-W. Hammer, and U. van Kolck, Nuclear Physics A 809, 171 (2008).



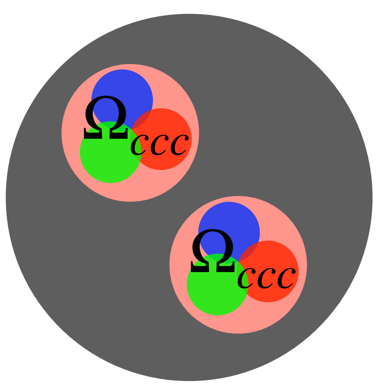


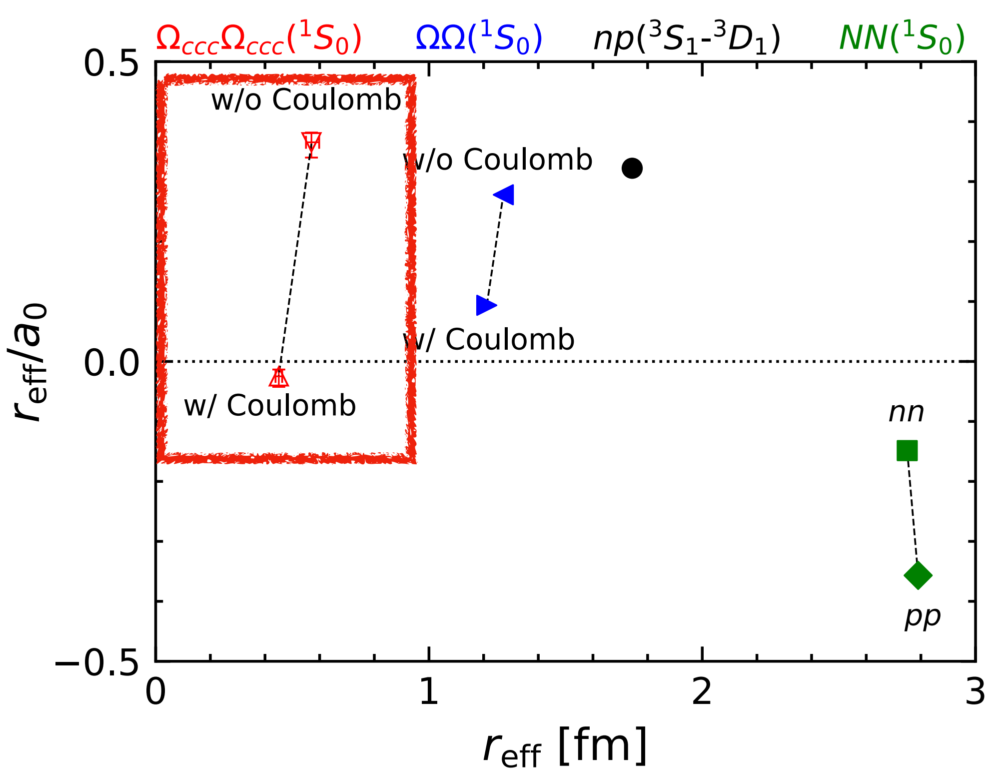
resonance (w/ Coulomb)

↑

bound (w/o Coulomb)

- $\Omega_{ccc}^{++}\,\Omega_{ccc}^{++}$ (HAL QCD) Y. Lyu, H.Tong, *et al*. [HAL QCD Coll.], Phys. Rev. Lett. 127 (2021) 072003.





resonance (w/ Coulomb)

↑

bound (w/o Coulomb)

- $\Xi^-\alpha$: Coulomb assisted bound state E. Hiyama, M. Isaka, T. Doi, and T. Hatsuda, Phys. Rev. C 106, 064318 (2022).

→ Coulomb is important for near-threshold states!

Coulomb + short range systems

◉ Coulomb + short range interaction

H. A. Bethe, Phys. Rev. 76, 38-50 (1949).

R. Oppenheim Berger and Larry Spruch, Phys. Rev. 138, B1106-B1115 (1965).

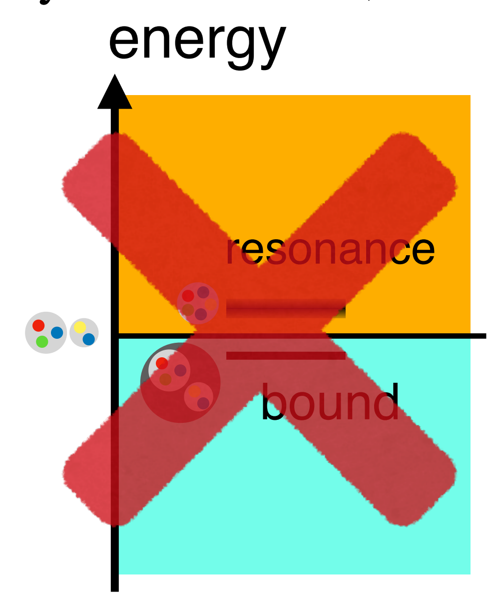
W. Domcke, Atom. Mol. Phys. 16, 359 (1983).

R. Higa, H.-W. Hammer, and U. van Kolck, Nuclear Physics A 809, 171 (2008).

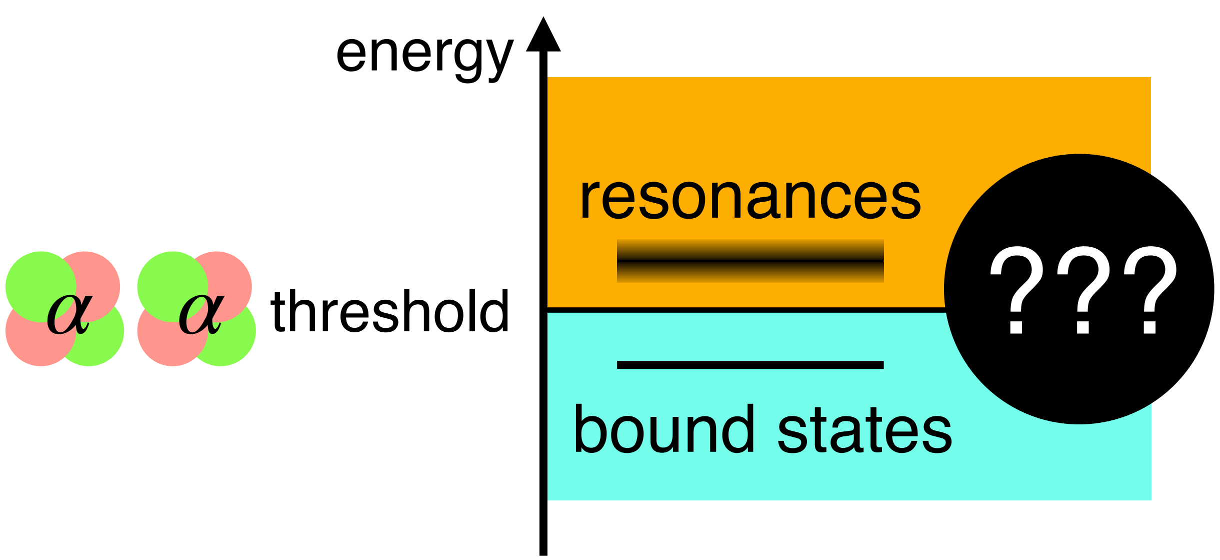
C. H. Schmickler, H.-W. Hammer, and A.G. Volosniev, Physics Letters B 798, 135016 (2019).

S. Mochizuki, and Y. Nishida, Phys. Rev. C 110 , 064001 (2024).

- low-energy behavior of scattering amplitude is different from that of short range interaction



◉ nature of near-threshold state with Coulomb + short range interaction?



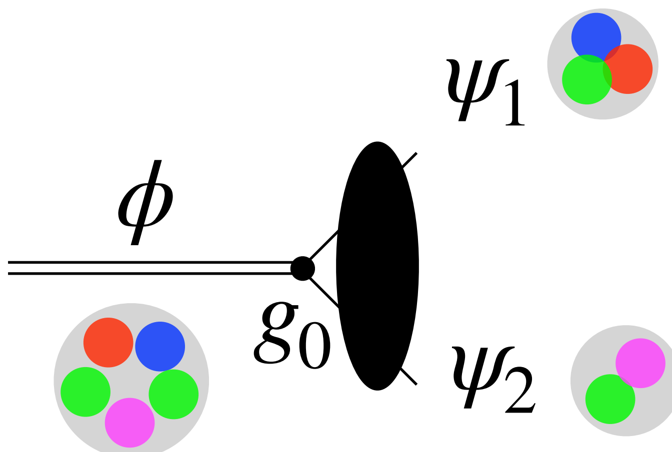
- two body
- small k region → s-wave

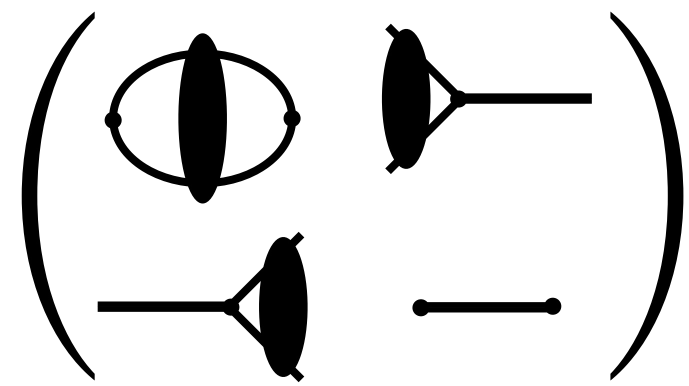
→ pole trajectories?
→ compositeness?

Coulomb + short range model

How does compositeness change across the threshold?

◉ model — W. Domcke, Atom. Mol. Phys. 16 359 (1983). R. Higa, H.-W. Hammer, and U. van Kolck, Nuclear Physics A 809 (2008).



$H =$ 

- pole condition (ERE with Coulomb contribution)

$$-\frac{1}{a_s} + \frac{r_e}{2}k^2 - ik \pm \frac{2}{a_B}\left[\log(-i a_B k) + \psi\left(1 \mp \frac{i}{a_B k}\right)\right] = 0$$

log cut

Bohr radius

$$a_B = \frac{\hbar c}{\mu c^2 Z_1 Z_2}$$

digamma function

$$\psi(z) = \frac{d}{dz}\log\Gamma(z)$$

Compositeness

S. Weinberg, Phys. Rev. 137, 672–678 (1965);
Y. Kamiya and T. Hyodo, PTEP 2017; Phys. Rev. C 93, 035203 (2016).

◉ expression of compositeness

$$X = 1 - \frac{1}{1 - \frac{d}{dE}\Sigma(E)} = \left[1 - \frac{r_e}{R^C}\right]^{-1}$$

self energy

c.f. only short-range interaction

$$X = \left[1 - \frac{r_e}{R}\right]^{-1}$$

R^C : radius with Coulomb contribution

$$R^C = = -\frac{1}{k}\left[2i\left(\frac{1}{a_B k}\right)^2 \psi_1\left(\mp i\frac{1}{a_B k}\right) + i - 2\left(\mp\frac{1}{a_B k}\right)\right]$$

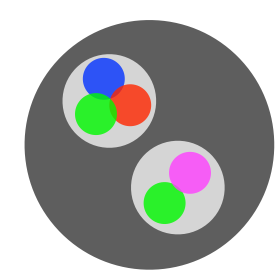
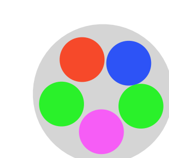
◉ interpretation scheme of X

extension of Ref.

I. Matuschek, V. Baru, F.K. Guo, C. Hanhart, Eur. Phys. J. A 57(3), 101 (2021).

^{8}Be & Ω_{ccc} dibaryon ⟶ resonance

$$X_C = \sqrt{\frac{1}{1 + \left| -\frac{1}{\Sigma'^2} + \frac{2}{\Sigma'} \right|}}$$

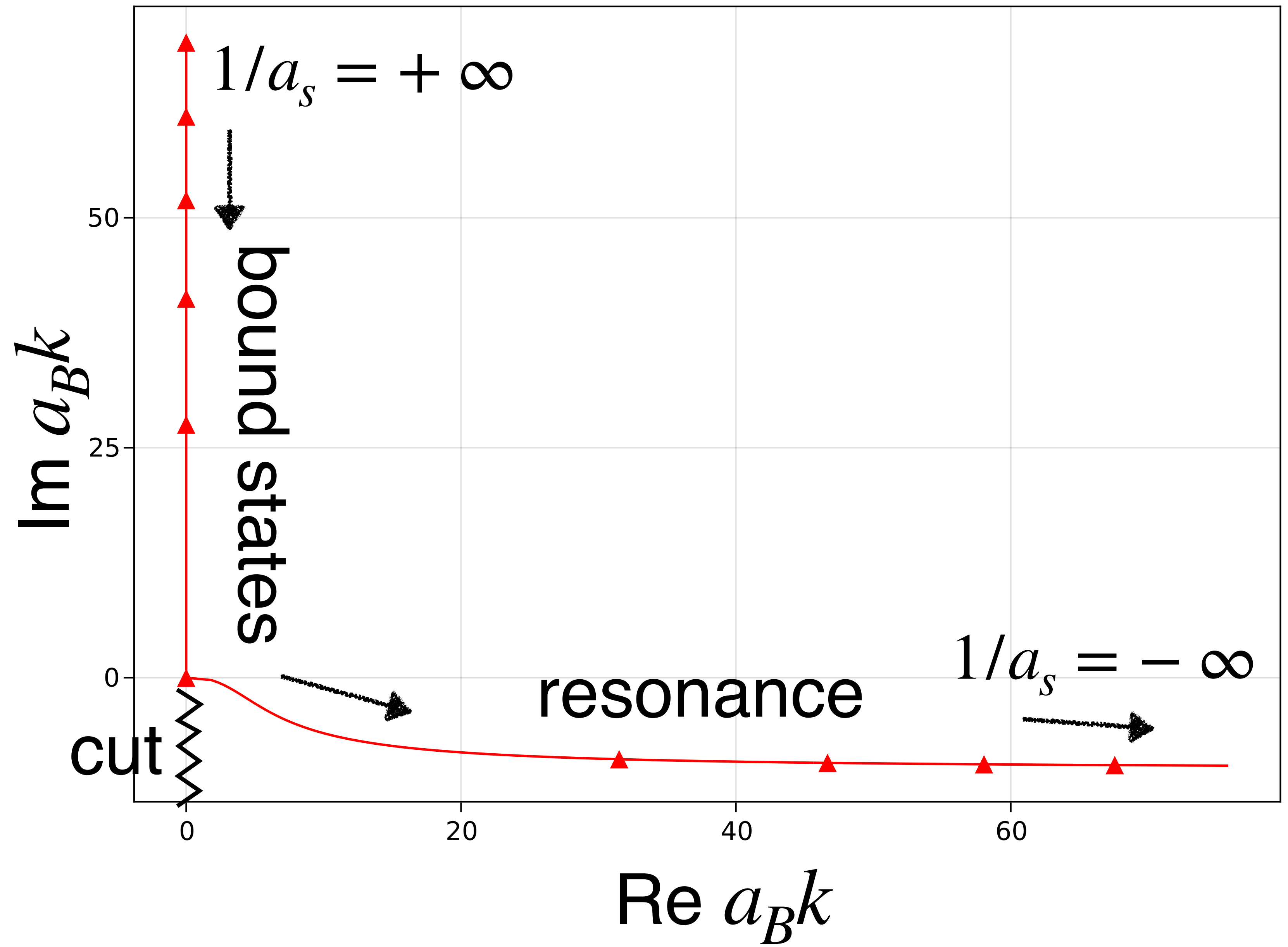
$0 \leq X_C \leq 1$ ⟶  or 

Pole trajectory (repulsive Coulomb)

◉ pole trajectory in complex momentum k plane

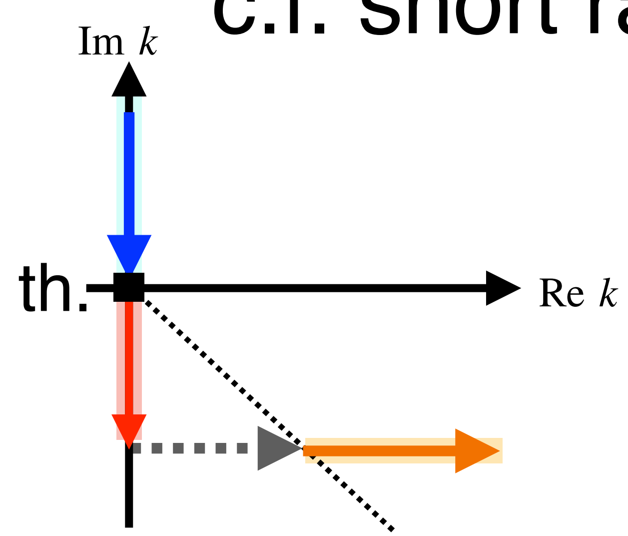
- varying Coulomb scattering length a_s with fixed r_e and a_B

→ pole position (eigenmomentum) moves



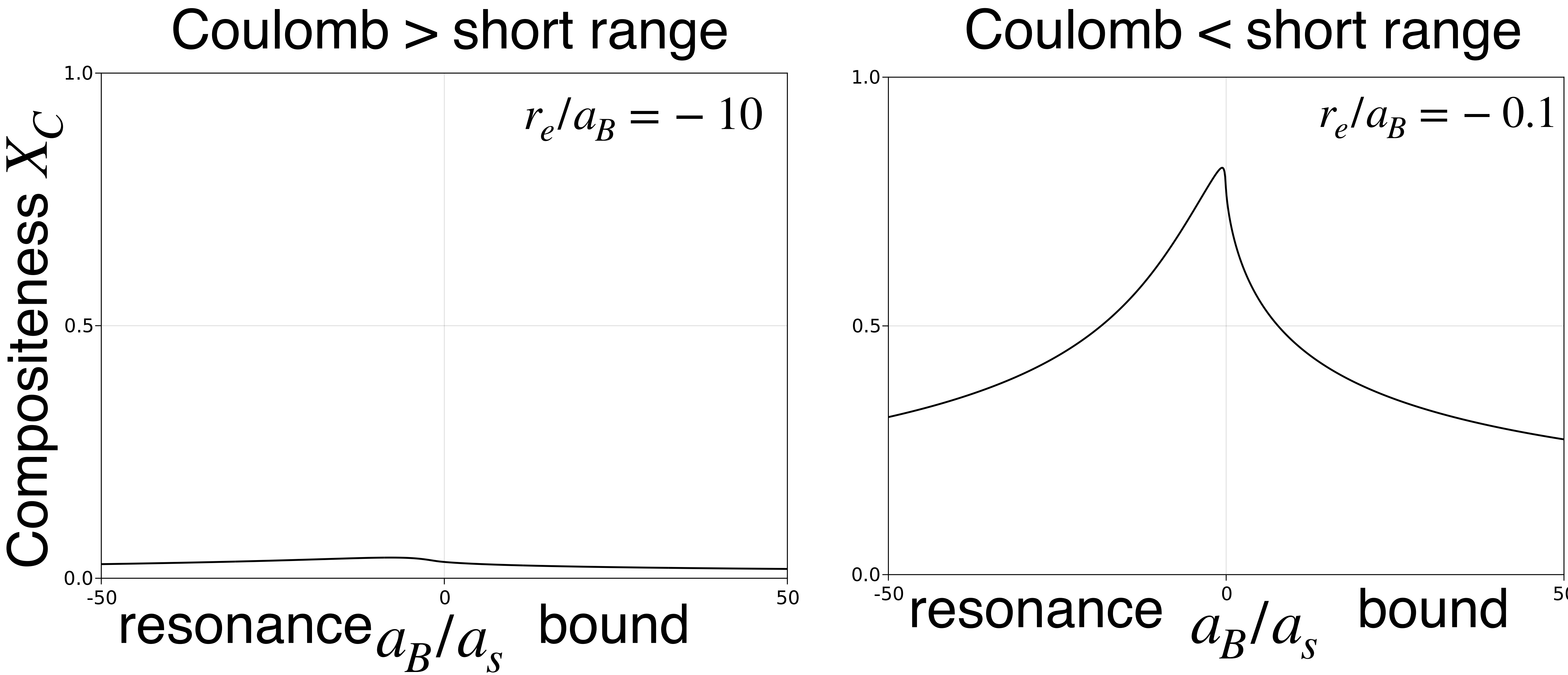
- b.s directory goes to resonance

c.f. short range



- no virtual states

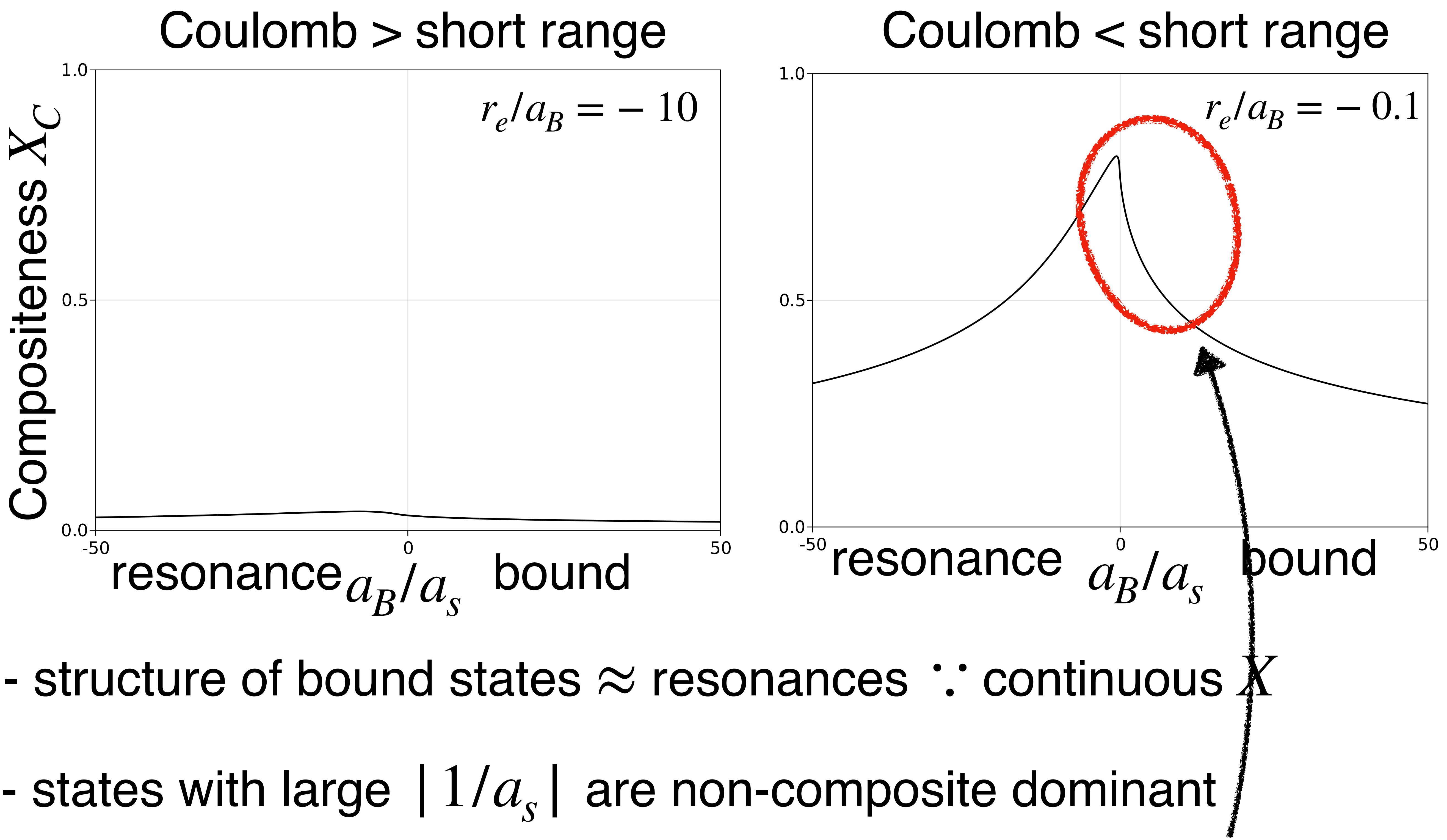
Compositeness (repulsive Coulomb)



- structure of bound states ≈ resonances ∵ continuous X

- states with large $|1/a_s|$ are non-composite dominant

Compositeness (repulsive Coulomb)



- structure of bound states ≈ resonances ∵ continuous X

- states with large $|1/a_s|$ are non-composite dominant

- remnant of short range universality in $|r_e| \ll |a_B|$ case

Application to near-th. states

	System	a_s^C (fm)	r_e^C (fm)	$\|r_e^C/a_B\|$
repulsive	pp	-7.82	2.83	0.049
	^{8}Be $(\alpha\alpha)$	-1.80×10^3	1.083	0.30
	$\Omega^-\Omega^-$	12.93	1.21	0.038
	$\Omega_{ccc}^{++}\Omega_{ccc}^{++}$	-19	0.45	0.16
attractive	$\Xi^-\alpha$	-12.6	6.66	0.48
	Ω^-p	3.2	1.2	0.029

pp X. Kong and F. Ravndal, Phys. Lett. B 450, 320 (1999)

^{8}Be R. Higa, H. W. Hammer, and U. van Kolck, Nucl. Phys. A 809, 171 (2008)

$\Omega\Omega$ and Ωp Y. Lyu, private communication

$\Omega_{ccc}\Omega_{ccc}$ Y. Lyu, H. Tong, T. Sugiura, S. Aoki, T. Doi, T. Hatsuda, J. Meng, and T. Miyamoto, Phys. Rev. Lett. 127, 072003 (2021)

$\Xi\alpha$ Y. Kamiya, private communication

Application to near-th. states

	System	Eigenenergy (MeV)	Property	X	X_C
rep	pp	$-0.14 - 0.47i$	V w/ width	$0.85 - 0.13i$	0.81
	^{8}Be $(\alpha\alpha)$	$0.083 - 8.3 \times 10^{-7}i$	R	$7.9 - 0.0013i$	0.71
	$\Omega^-\Omega^-$	-0.85	B	1.5	0.81
......	$\Omega_{ccc}^{++}\Omega_{ccc}^{++}$	$1.3 - 0.17i$	R	$1.5 - 0.2i$	0.79
att	$\Xi^-\alpha$	-0.22	B	1.1	1.0
	Ω^-p	-1.07	B	1.2	0.87

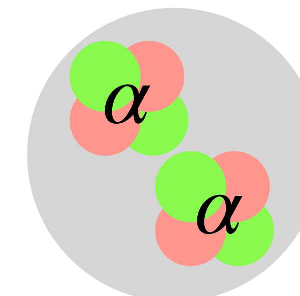
- a_s, r_e and a_B ⟶ eigenenergy using ERE
- pp: virtual with width ⟵ nn virtual state
- ^{8}Be: cluster dominant structure ⟵ experiment & few-body calculations

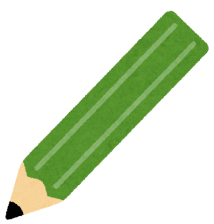
R. B. Wiringa, S. C. Pieper, J. Carlson, and V. R. Pandharipande, Phys. Rev. C 62, 014001 (2000).
T. Otsuka, T. Abe, T. Yoshida, Y. Tsunoda, N. Shimizu, N. Itagaki, Y. Utsuno, J. Vary, P. Maris, and H. Ueno, Nature Commun. 13, 2234 (2022).

- $\Omega\Omega$ and $\Omega_{ccc}\Omega_{ccc}$: consistent with lattice calculation
- $\Xi\alpha$: purely composite - Ωp: consistent with lattice calc.

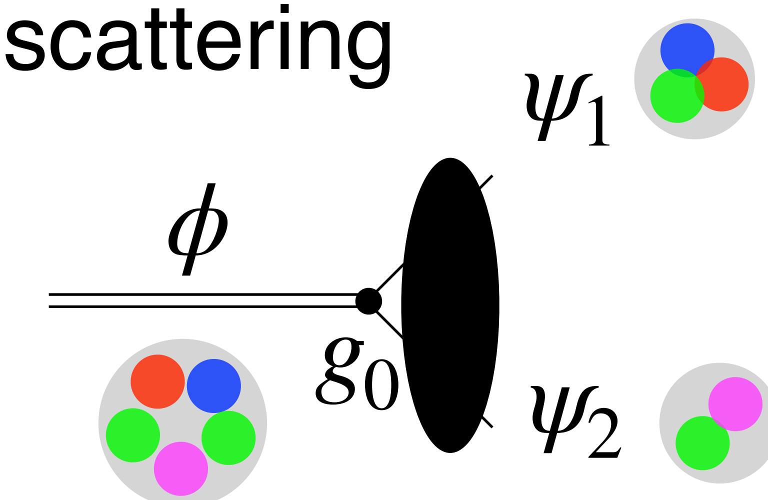
Summary

near-threshold bound states & resonances
with **Coulomb + short range** interaction





- bare state which couples to Coulomb scattering
- pole condition ← a_s, r_e, a_B



- repulsive Coulomb

bound → resonance (does not become virtual states)

nature of b.s. ≈ nature of resonance

if Coulomb < s.r., remnant of s.r. universality can be seen

- application to near-threshold states

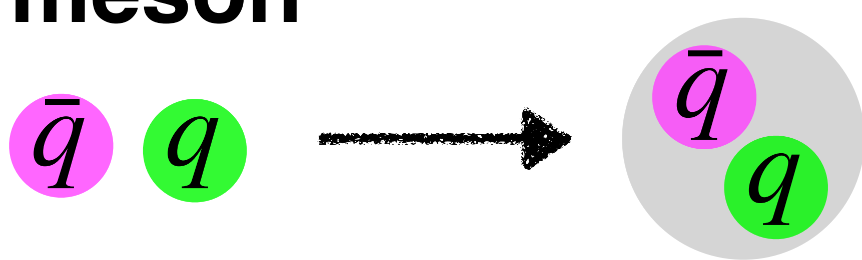
Back up

Exotic hadron

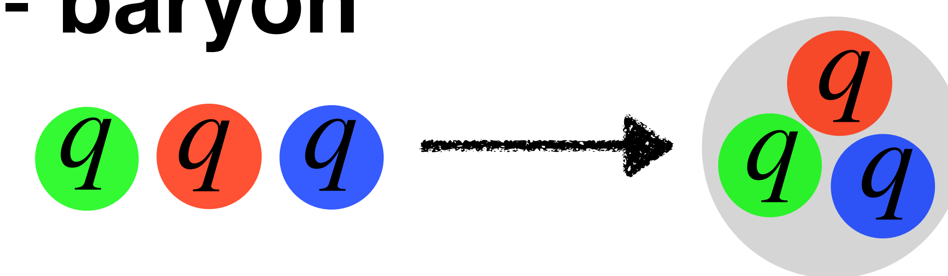
quark q, gluon → hadron
strong interaction

◉ ordinary hadron

- **meson**



- **baryon**

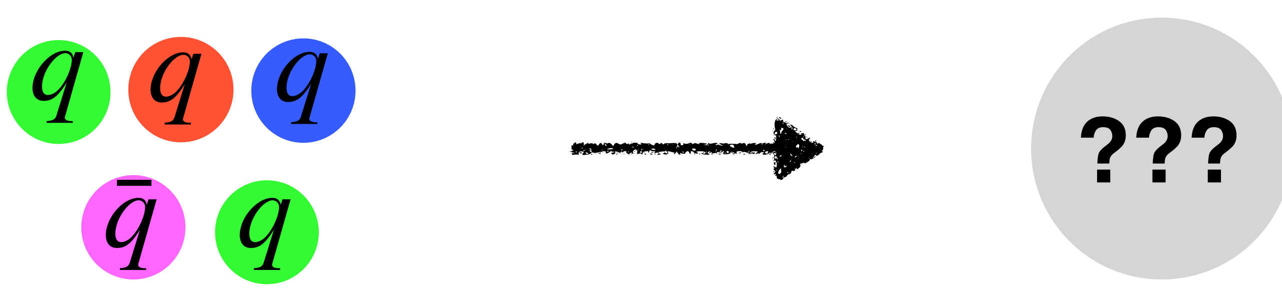


- ~400 kinds have been observed so far

Particle Data Group, S. Navas et al., Phys. Rev. D 110, 030001 (2024).

◉ exotic hadron

- composed of four or more quarks

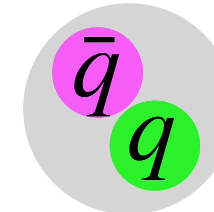
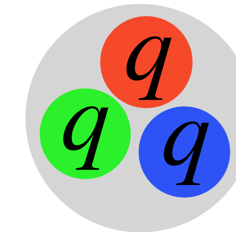


internal structure?

- **rarely** observed ← Why?
- hint for understanding low-energy phenomena in QCD

Exotic hadron

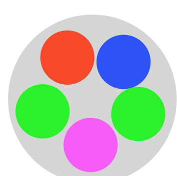
A. Hosaka, T. Iijima, K. Miyabayashi, Y. Sakai, and S. Yasui, PTEP 2016, 062C01 (2016)

- ordinaly hadron: $q\bar{q}$ (meson) or qqq (baryon)  or 

- other than $q\bar{q}$ or qqq → **exotic hadron** → structure?

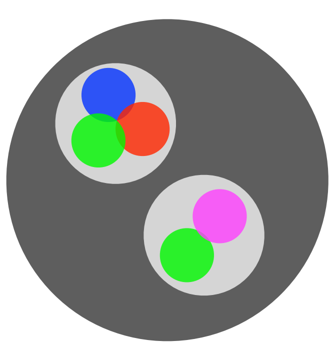
◉ <u>possible structure</u> F-K. Guo, C. Hanhart, Ulf-G. Meißner, Q. Wang, Q. Zhao, and B-S. Zou, RevModPhys.90.015004 (2018).

- **multiquarks**: quarks → hadron



size $\lesssim$ 1 fm (**compact**)

- **hadronic molecule**: quarks → hadrons (subunit) → hadron



size $\gtrsim$ 1 fm (spatially large radious)

e.g. deuteron (pn molecule), $R = 4.32$ fm

→ **superposition** of possible components

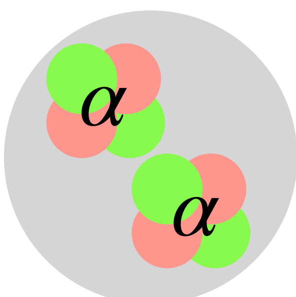
internal structure ← investigate weight of each component

Clustering phenomena

- near-threshold states ⟶ clustering phenomena

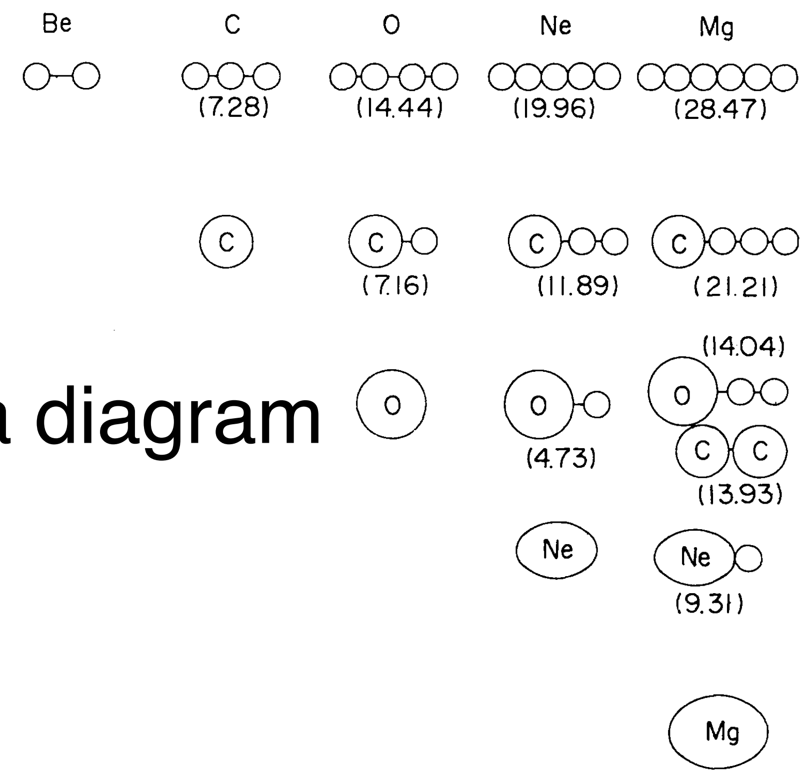
- α clustering phenomena

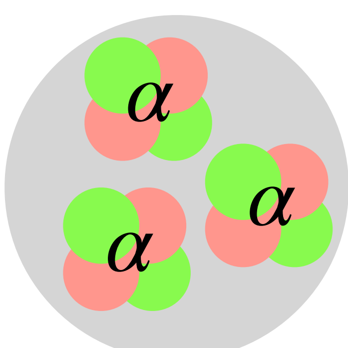
K. Ikeda, and N. Takizawa, and H. Horiuchi, Prog. Theor. Phys. Suppl. E68, 464-475 (1968).



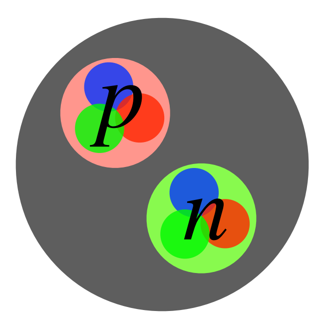
^{8}Be nuclei

Ikeda diagram



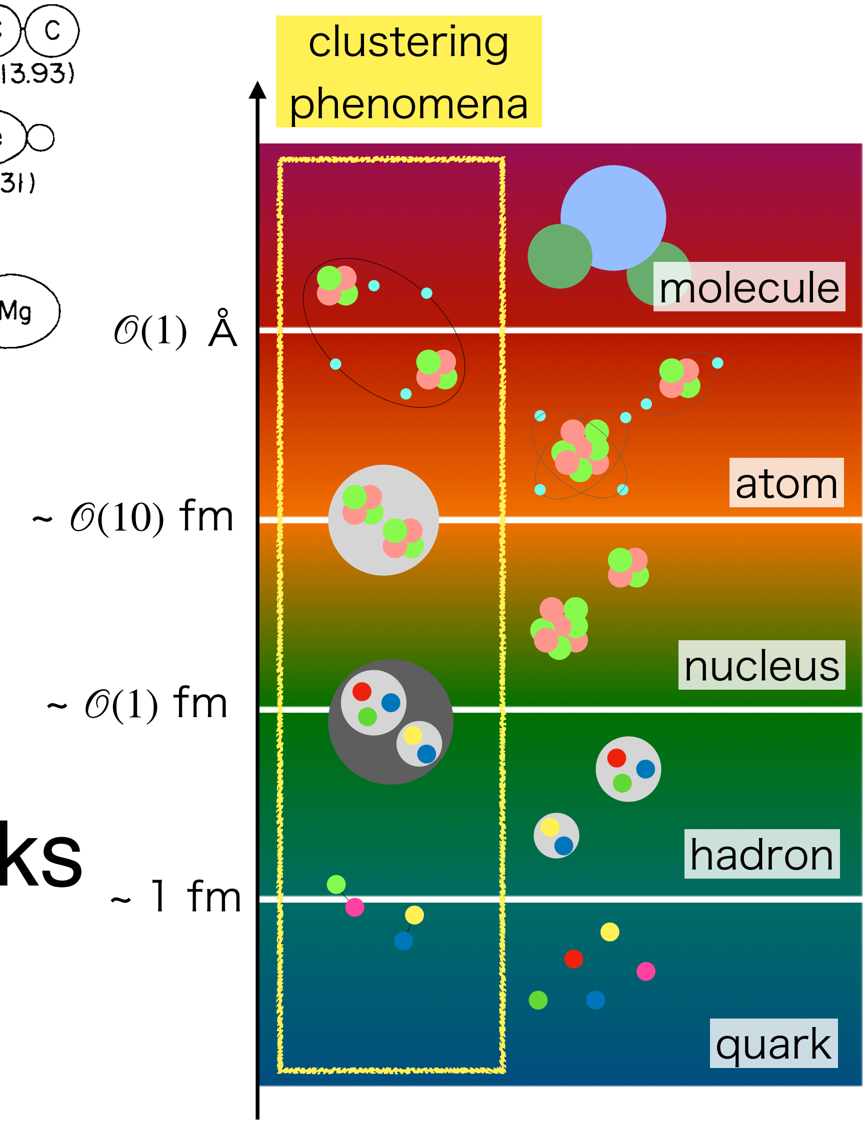


^{12}C Hoyle state



- deuteron $B = 2.22$ MeV

hadronic molecule ⟶ clustering of quarks

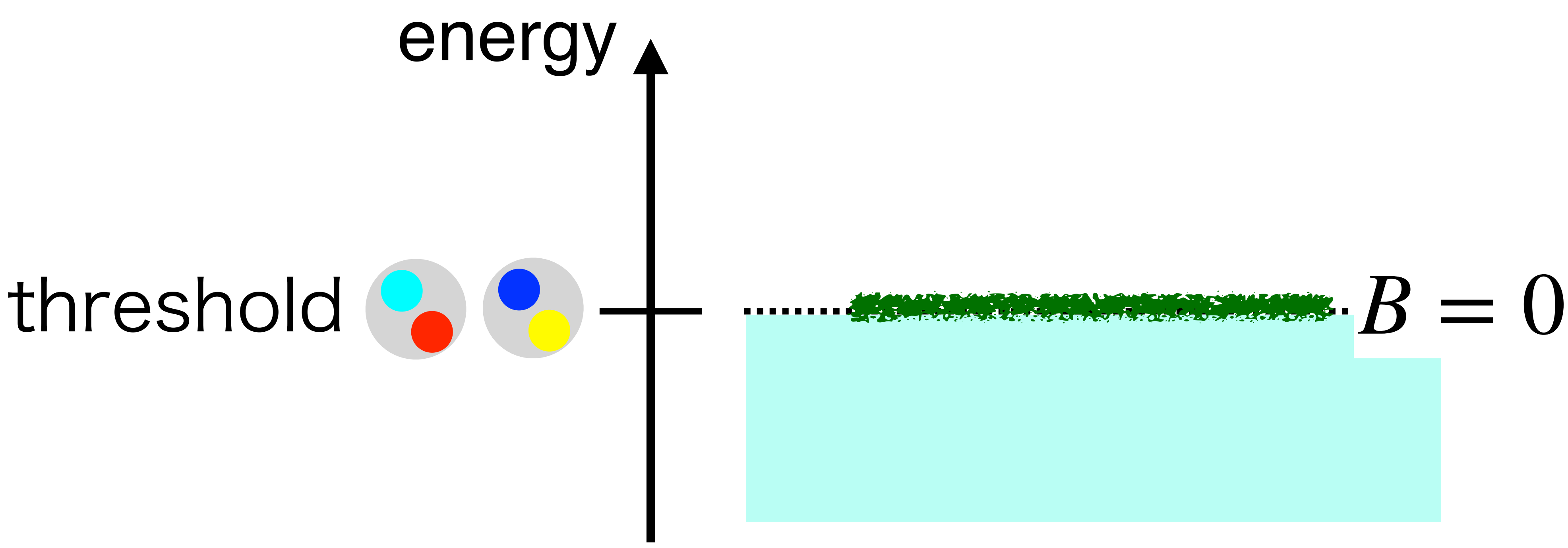


- clustering phenomena: **universal**!

Low-energy universality

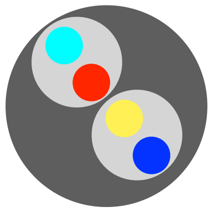
- near-threshold energy region ← universal phenomena

E. Braaten and H.-W. Hammer, Phys. Rept. **428**, 259 (2006); P. Naidon and S. Endo, Rept. Prog. Phys. **80**, 056001 (2017).



- binding energy $B = 0$ (exactly at threshold)

→ that state is **always** completely composite ($X = 1$)



compositeness is model-**independently** determined

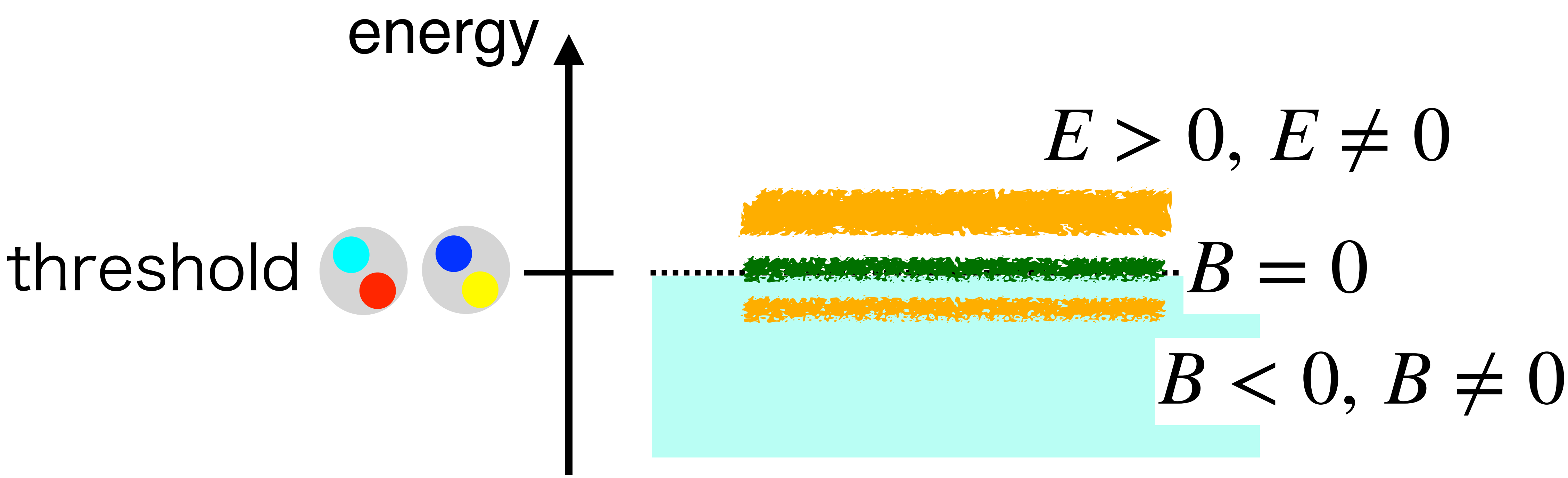
T. Hyodo, Phys. Rev. C **90**, 055208 (2014) .

← this work

Low-energy universality

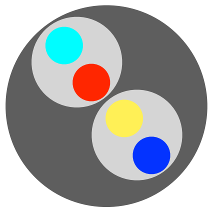
- near-threshold energy region ← universal phenomena

E. Braaten and H.-W. Hammer, Phys. Rept. **428**, 259 (2006); P. Naidon and S. Endo, Rept. Prog. Phys. **80**, 056001 (2017).



- binding energy $B = 0$ (exactly at threshold)

→ that state is **always** completely composite ($X = 1$)



compositeness is model-**independently** determined

T. Hyodo, Phys. Rev. C **90**, 055208 (2014) .

- structure of **near**-threshold states ($B \neq 0$)? ← this work

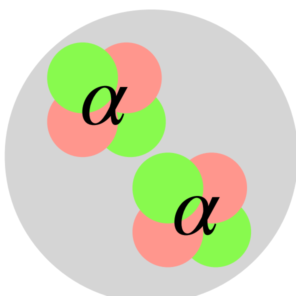
similar to $B = 0$ state? $X \sim 1$?

Near-threshold states

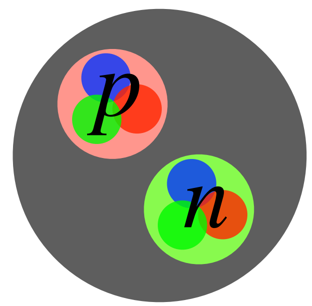
◉ Are near-threshold states composite dominant ($X \sim 1$)?

- **empiricaly** : Yes! "threshold energy rule"

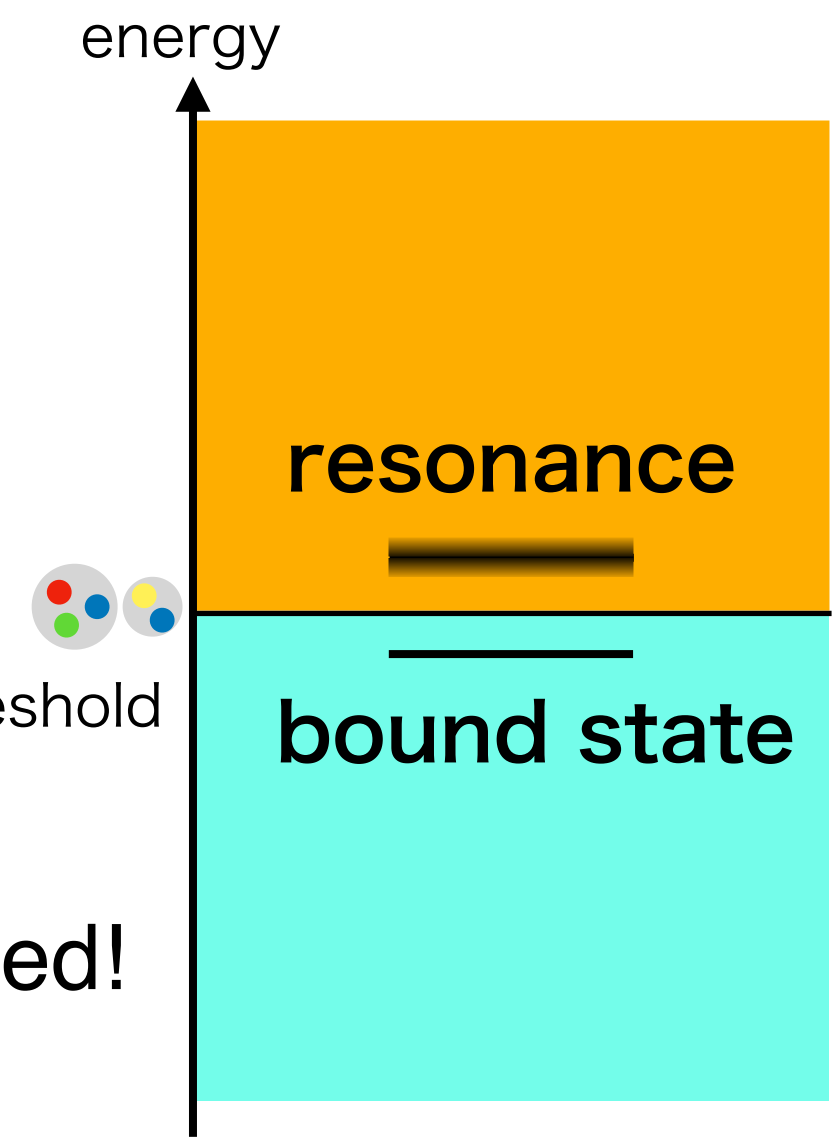
K. Ikeda, and N. Takizawa, and H. Horiuchi, Prog. Theor. Phys. Suppl. E68, 464-475 (1968).



^{8}Be nuclei



deuteron



- **theoretically**

Even non-composite dominant shallow bound states ($X \sim 0$) can be realized!

T. Hyodo, Phys. Rev. C **90**, 055208 (2014) ;
C. Hanhart, J. R. Pelaez, and G. Rios, Phys. Lett. B **739**, 375 (2014).

→ Why do we rarely observe such a state?

structure of **resonances** slightly above threshold?

→ provide theoretical foundation of threshold energy rule

Compositeness

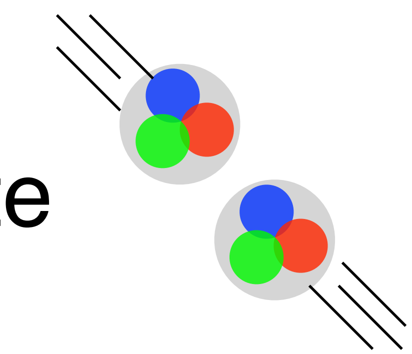
Weinberg, S. Phys. Rev. 137, 672–678 (1965);
T. Hyodo, Int. J. Mod. Phys. A 28, 1330045 (2013);
T. Kinugawa, T. Hyodo, arXiv:2411.12285 [hep-ph] (accepted in EPJ A).

- decomposing Hamiltonian into free part and remainder

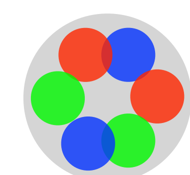
$$\hat{H} = \hat{H}_0 + \hat{V}$$

free Hamiltonian

$$\hat{H}_0 |\boldsymbol{p}\rangle = \frac{\boldsymbol{p}^2}{2\mu} |\boldsymbol{p}\rangle$$ free scattering state



$$\hat{H}_0 |\phi\rangle = \nu_0 |\phi\rangle$$ bare discrete state



- compositeness X, elementarity Z

$$X = \int d\boldsymbol{p} \; |\langle \boldsymbol{p} | B \rangle|^2, \quad Z = |\langle \phi | B \rangle|^2$$

overlaps between bound state $|B\rangle$ and $|\boldsymbol{p}\rangle$ or $|\phi\rangle$

- decomposition is not unique! ⟶ **model dependence** of X

S. Weinberg, Phys. Rev. 130, 776 (1963); D. Luri´e and A. J. Macfarlane, Phys. Rev. 136, B816 (1964).

Compositeness

◉ model calculation

T. Hyodo, D. Jido, and A. Hosaka, Phys. Rev. C **85**, 015201 (2012);
F. Aceti and E. Oset, Phys. Rev. D **86**, 014012 (2012).

$$T = \frac{1}{V^{-1} - G}$$

V : effective interaction

G : loop function

residue of scattering amplitude g

$$X = -g^2 G'(E)\Big|_{E=-B} = \frac{G'(E)}{G'(E) - [V^{-1}(E)]'}\Bigg|_{E=-B}$$

$$\alpha'(E) = d\alpha/dE$$

Y. Kamiya and T. Hyodo, PTEP 2017, 023D02 (2017).

g^2 : model independent ⟵ $T_{\rm on}(-B)$ (observable)

$G(E)$: model dependent ⟵ cutoff dependent

History of compositeness

- Weinberg’s work (1960s) Weinberg, S. Phys. Rev. 137, 672–678 (1965) etc.

  deuteron is not an elementary particle ⟵ weak-binding relation

  Z : field renormalization factor

- application to exotic hadrons (2000s-)

  X: “compositeness”

  generalization to unstable states

  - with spectral function V. Baru, J. Haidenbauer, C. Hanhart, Y. Kalashnikova, A.E. Kudryavtsev, Phys. Lett. B 586, 53–61 (2004) etc.
  - with effective range expansion T. Hyodo, Phys. Rev. Lett. **111**, 132002 (2013) etc.
  - with effective field theory Y. Kamiya and T. Hyodo, PTEP 2017, 023D02 (2017) etc.

  application to …

  - $f_0(980)$, $a_0(980)$ Y. Kamiya and T. Hyodo, PTEP 2017, Phys. Rev. C 93, 035203 (2016); T. Sekihara, S. Kumano, Phys. Rev. D 92, 034010 (2015) etc.
  - $\Lambda(1405)$ T. Sekihara, T. Hyodo, Phys. Rev. C 87, 045202 (2013) ; Z.H. Guo, J.A. Oller, Phys. Rev. D 93, 096001 (2016) etc.
  - nuclei & atomic systems T. Kinugawa, T. Hyodo, Phys. Rev. C 106, 015205 (2022) etc.

Weak-binding relation

S. Weinberg, Phys. Rev. 137, 672–678 (1965).

$$X = \frac{a_0}{2R - a_0} + \mathcal{O}\left(\frac{R_{\mathrm{typ}}}{R}\right)$$

a_0 : scattering length

R_{typ} : typical length scale in system

$R = 1/\sqrt{2\mu B}$

- for weakly bound states, $R \gg R_{\mathrm{typ}}$

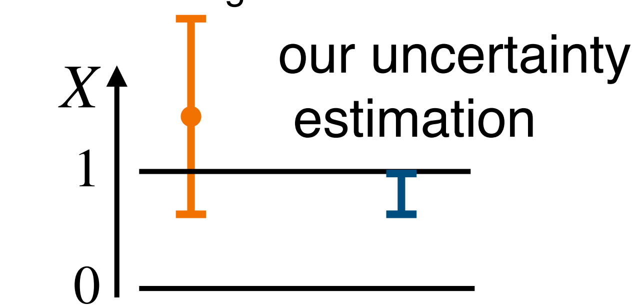
compositeness ⟵ observables (a_0, B)

Y. Li, F.-K. Guo, J.-Y. Pang, and J.-J. Wu, Phys. Rev. D 105, L071502 (2022);
J. Song, L. R. Dai, and E. Oset, Eur. Phys. J. A 58, 133 (2022);
M. Albaladejo, J. Nieves, Eur. Phys. J. C 82, 724 (2022);
T. Kinugawa, T. Hyodo, Phys. Rev. C 106, 015205 (2022);
Z. Yin and D. Jido, Phys. Rev. C 110, no.5, 055202 (2024).

◉ range correction

compositeness of deuteron $X \sim 1.7 > 1$

⟶ important to consider effective range



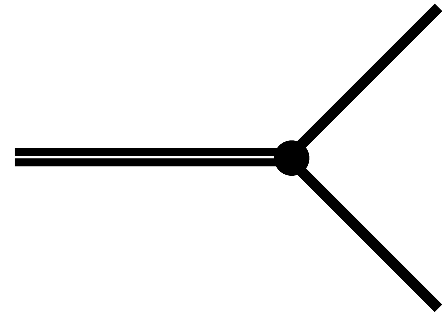
- our work : range correction ⟵ uncertainty estimation

compositeness of deuteron : $0.74 \leq X \leq 1$

Pole trajectory (only w/ s.r.)

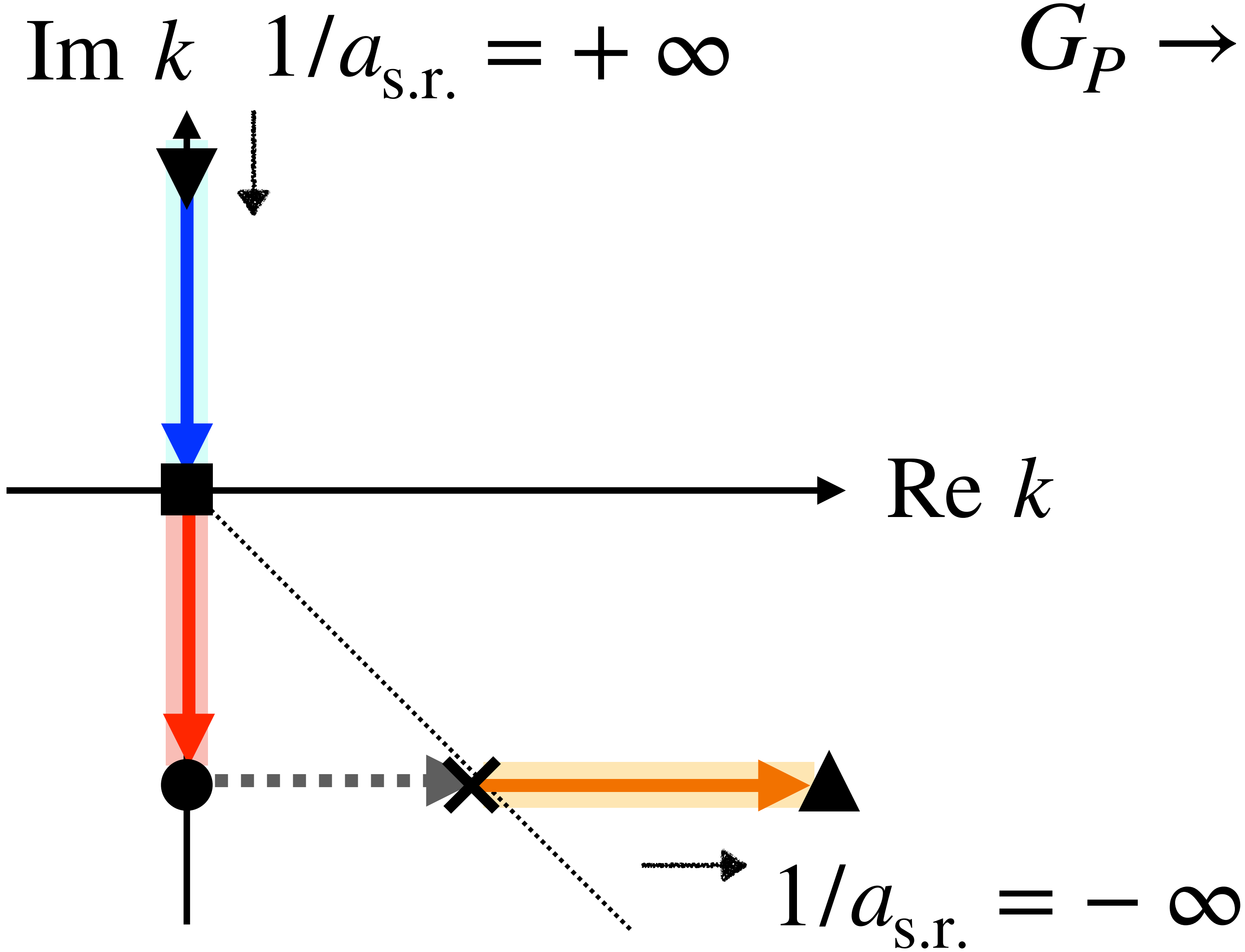
◉ <u>pole trajectory in complex momentum k plane (No Coulomb)</u>

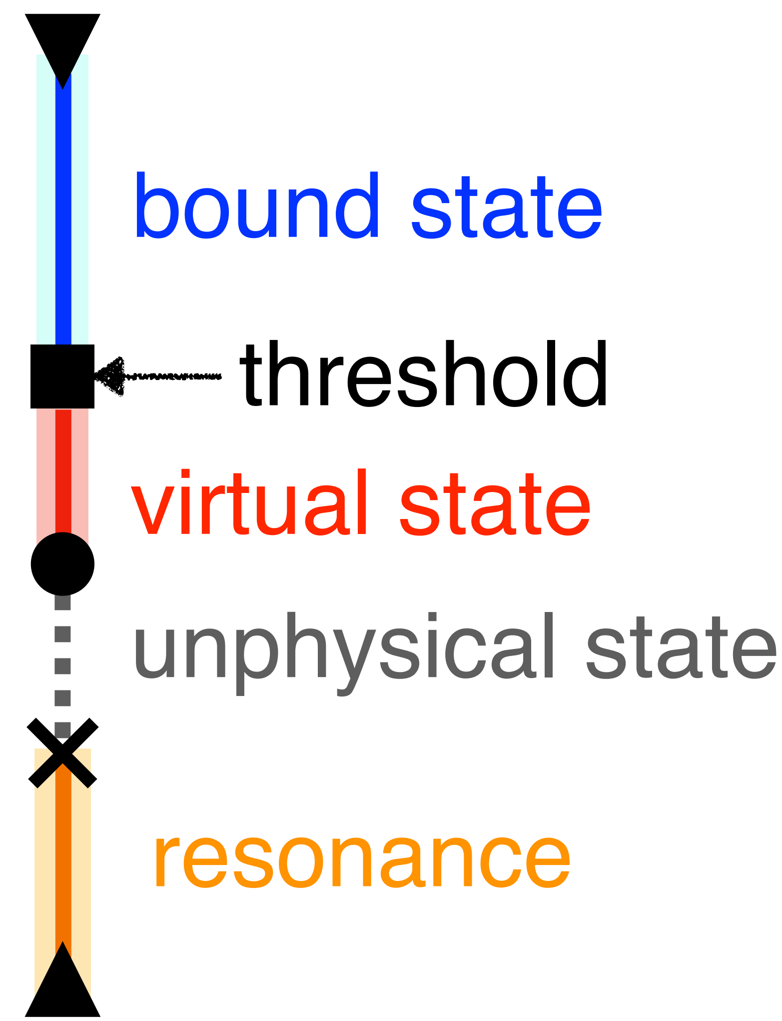
Q channel
⇔ bare state



P channel
⇔ free scattering

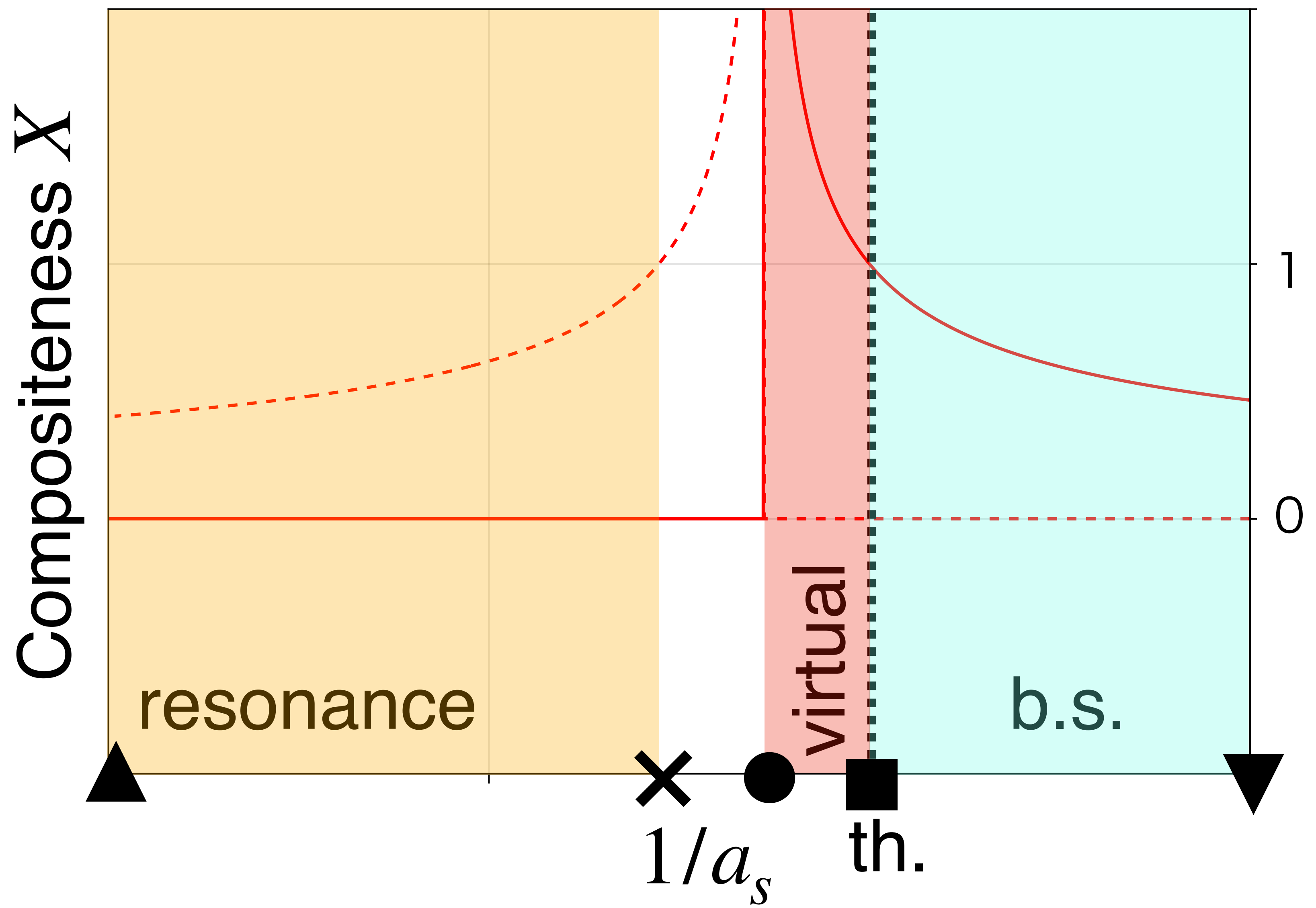
$G_P \to G^0$ free Green's function





T. Hyodo, Phys. Rev. Lett. **111**, 132002 (2013) .

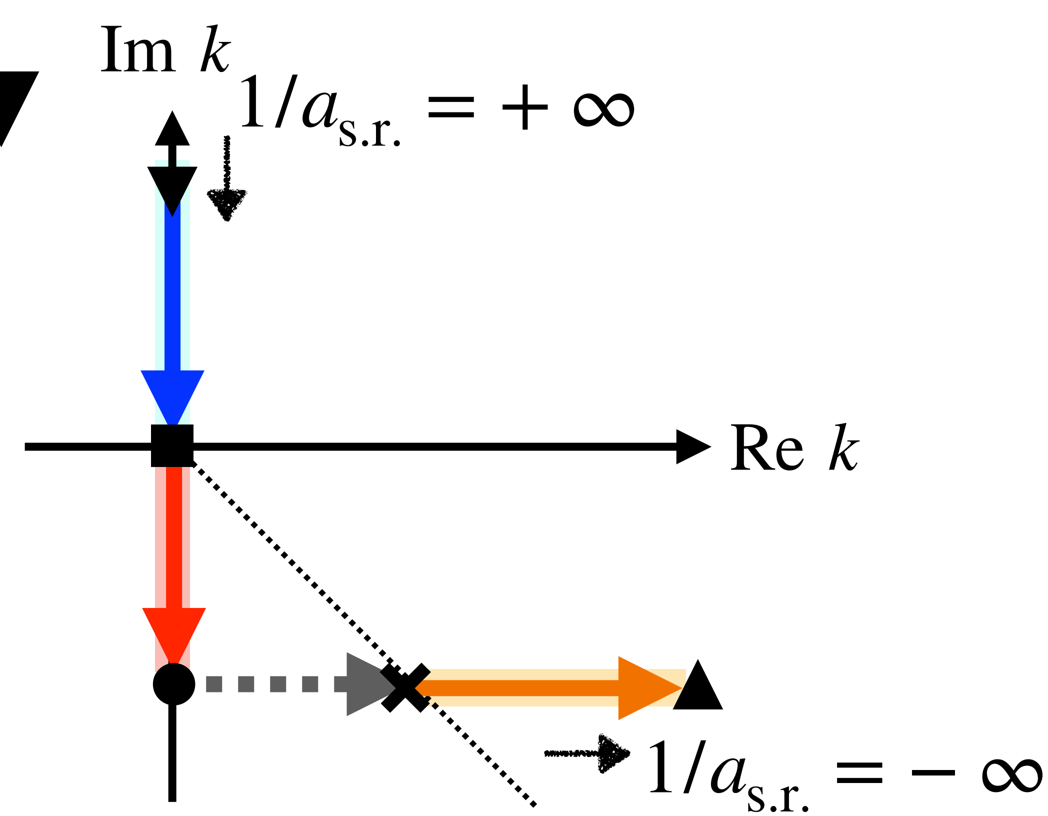
Compositeness (only w/ s.r.)



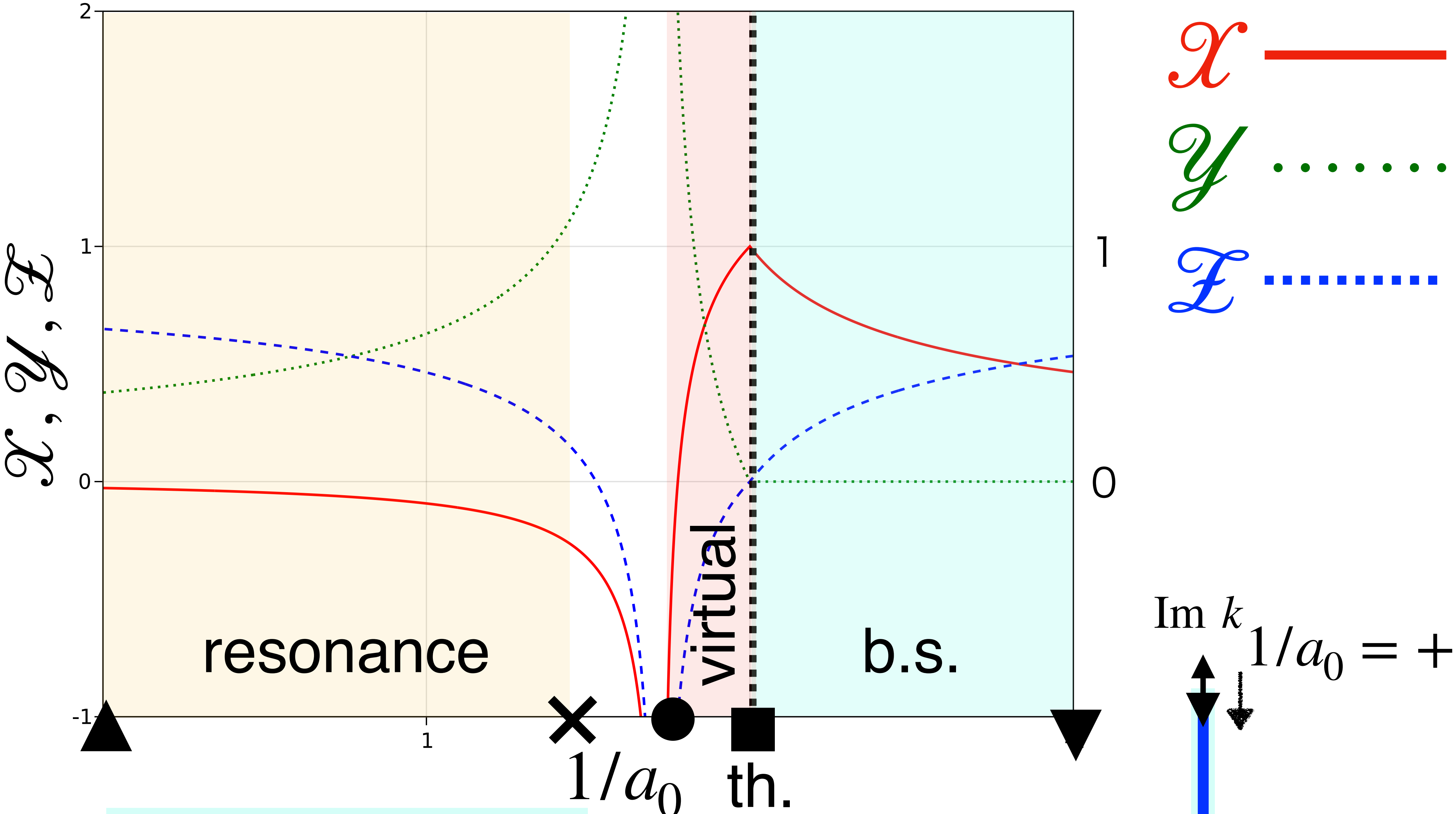
- b.s. : $0 \leq X \leq 1$
- virtual : $1 < X$

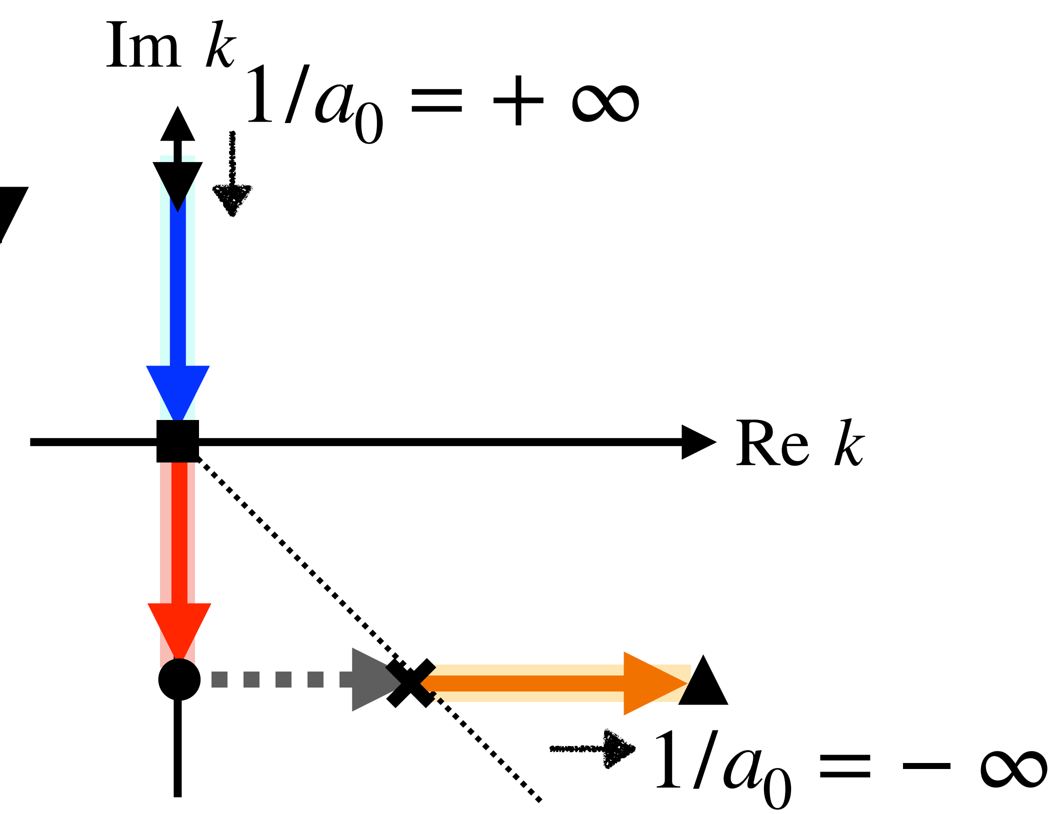
● divergence $X \to \infty$

- resonance : $\mathrm{Im}\ X \leq 1$



$\mathscr{X}, \mathscr{Y}, \mathscr{Z}$ from b.s. to resonance



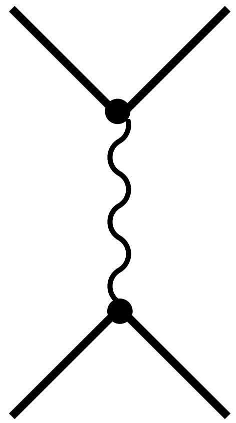


- b.s. : $0 \leq X \leq 1$
- virtual : $\mathscr{Z} < 0$ (non-interpretable)
- ● divergence $\mathscr{X}, \mathscr{Y}, \mathscr{Z} \to \infty$
- resonance : $\mathscr{Z}$ dominant

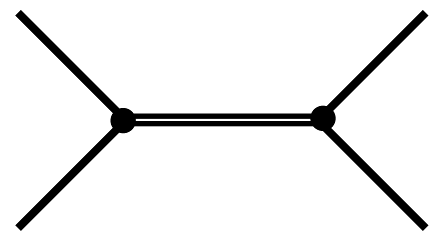
Coulomb+short range model

◉ model

Coulomb



short range (s.r.)



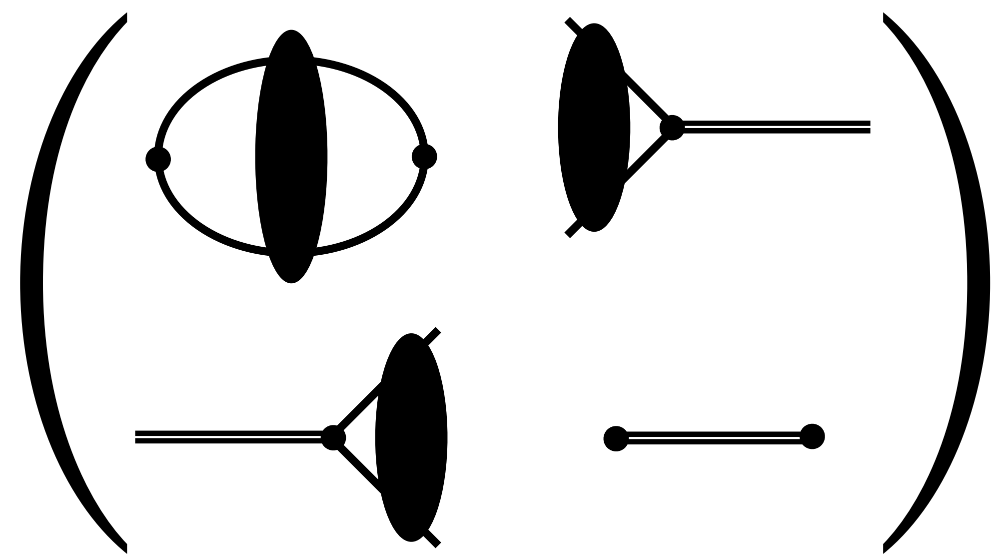
S. Weinberg, Phys. Rev. 137, 672–678 (1965);
V. Baru, J. Haidenbauer, C. Hanhart, Y. Kalashnikova, A.E. Kudryavtsev, Phys. Lett. B 586, 53–61 (2004);
T. Hyodo, Phys. Rev. C **90**, 055208 (2014) .

◉ Hamiltonian

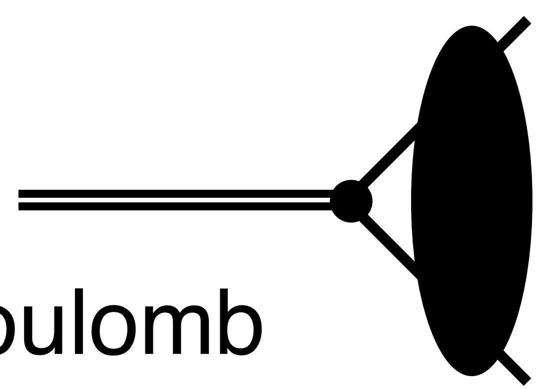
W. Domcke, Atom. Mol. Phys. 16 359 (1983);
H. Feshbach, Annals Phys. 19 287-313 (1962).

C. H. Schmickler, H.-W. Hammer, and A.G. Volosniev, Physics Letters B 798, 135016

$$\hat{H} = \begin{pmatrix} \hat{H}_{PP} & \hat{H}_{PQ} \\ \hat{H}_{QP} & \hat{H}_{QQ} \end{pmatrix} =$$



Q channel
⇔ b.s. w/o Coulomb



P channel
⇔ scattering w/ Coulomb

Coulomb+short range model

◉ Schrödinger equation $\hat{H}|\Psi\rangle = E|\Psi\rangle \quad |\Psi\rangle = \begin{pmatrix} |P\rangle \\ |Q\rangle \end{pmatrix}$

$$\hat{H}_{PP}|P\rangle + \hat{H}_{PQ}|Q\rangle = E|P\rangle$$
$$\hat{H}_{QQ}|Q\rangle + \hat{H}_{QP}|P\rangle = E|Q\rangle$$

◉ effective Hamiltonian (channel eliminating)

$$\hat{H}_{P\mathrm{ch}}|P\rangle = E|P\rangle \quad \hat{H}_{P\mathrm{ch}} = \hat{H}_{PP} + \hat{H}_{PQ}(E - \hat{H}_{QQ})^{-1}\hat{H}_{QP}$$

Coulomb+short range model

◉ Schrödinger equation $\hat{H}|\Psi\rangle = E|\Psi\rangle \quad |\Psi\rangle = \begin{pmatrix} |P\rangle \\ |Q\rangle \end{pmatrix}$

$$\hat{H}_{PP}|P\rangle + \hat{H}_{PQ}|Q\rangle = E|P\rangle$$
$$\hat{H}_{QQ}|Q\rangle + \hat{H}_{QP}|P\rangle = E|Q\rangle$$

◉ effective Hamiltonian (channel eliminating)

$$\hat{H}_{P\mathrm{ch}}|P\rangle = E|P\rangle \quad \hat{H}_{P\mathrm{ch}} = \hat{H}_{PP} + \hat{H}_{PQ}(E - \hat{H}_{QQ})^{-1}\hat{H}_{QP}$$

$$\hat{H}_{PP} = \hat{H}^0 + \hat{V}_P \qquad \hat{H}_{PQ}(E - \hat{H}_{QQ})^{-1}\hat{H}_{QP} = \hat{V}_Q$$

$$\longrightarrow \hat{H}_{P\mathrm{ch}} = \hat{H}^0 + (\hat{V}_P + \hat{V}_Q)$$

$\hat{H}^0$: free Hamiltonian

$\hat{V}_P$: pure Coulomb interaction

$\hat{V}_Q$: short range interaction

Coulomb+short range model

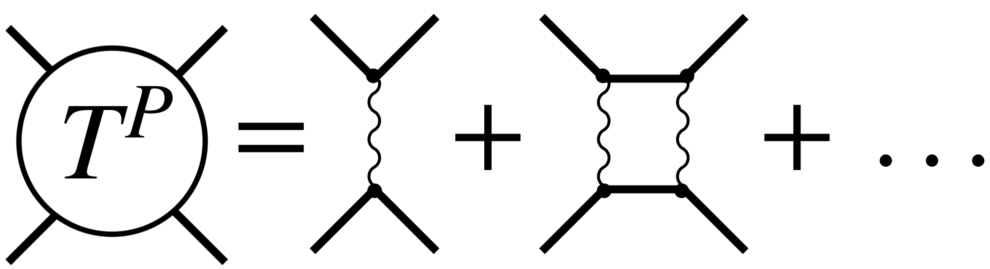
◉ *T*-matrix

H. Feshbach, Annals Phys. 19 287-313 (1962); W. Domcke, Atom. Mol. Phys. 16 359 (1983); R. Higa, H.-W. Hammer, and U. van Kolck, Nuclear Physics A 809 (2008).

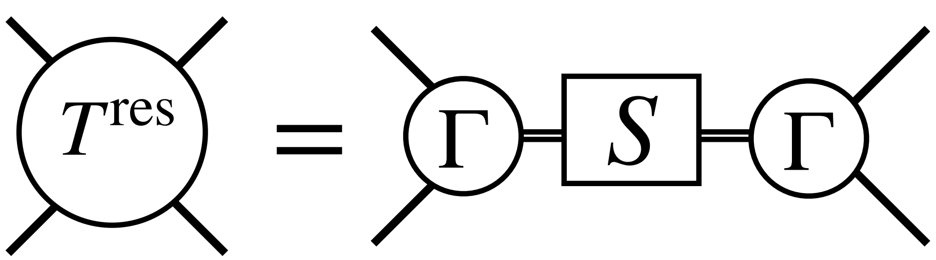
Lippmann-Schwinger eq. : $\hat{T} = [(\hat{V}_P + \hat{V}_Q)^{-1} - \hat{G}^0]^{-1}$

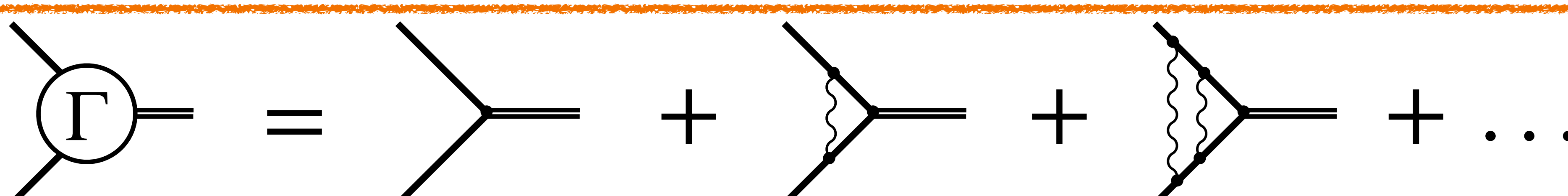
⇔ Feshbach method : $\hat{T} = \hat{T}^P + \hat{T}^{\rm res}$

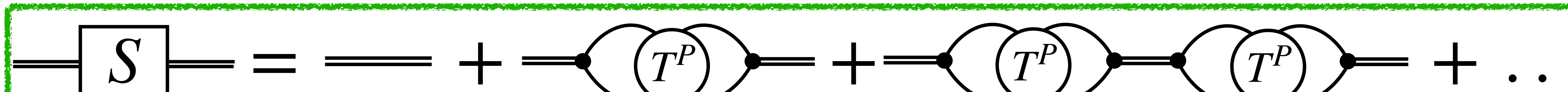
$$\hat{T}^P = [\hat{V}_P^{-1} - \hat{G}^0]^{-1}, \quad \hat{T}^{\rm res} = \hat{T}^P \hat{V}_P^{-1} [\hat{V}_Q^{-1} - \hat{G}_P]^{-1} \hat{V}_P^{-1} \hat{T}^P$$



pure Coulomb



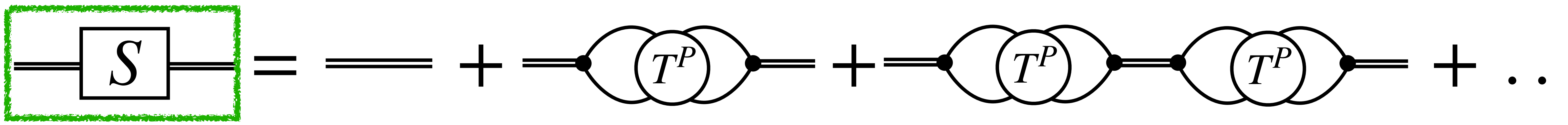


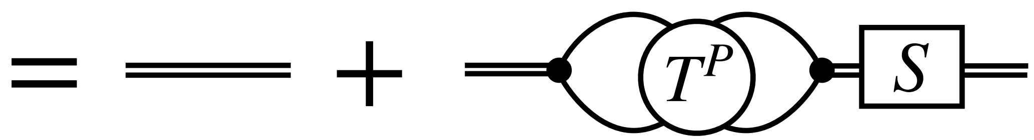


Coulomb+short range model

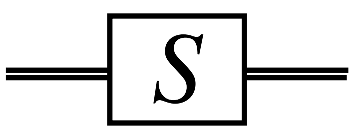
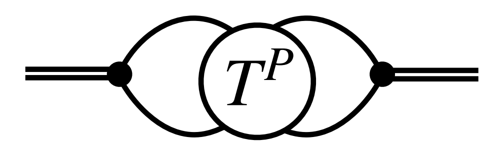
◉ pole condition $\hat{T} = \hat{T}^P + \hat{T}^{\rm res}$

pole of $T(k,k')$ $\Leftrightarrow$ pole of $T^{\rm res}(k,k')$ $\Leftrightarrow$ $[V_Q^{-1} - G_P]^{-1} = \infty$





$H_{QP}G_PH_{PQ}$

 $= S(E)$,  $= (E-\varepsilon_d)^{-1}$,  $= F(E)$

$$\longrightarrow S(E) = (E-\varepsilon_d)^{-1} + (E-\varepsilon_d)^{-1}F(E)S(E)$$

$$= [E - \varepsilon_d - F(E)]^{-1}$$

$\longrightarrow$ pole condition : $E - \varepsilon_d - F(E) = 0$

bare state energy   self energy

Coulomb+short range model

◉ self energy $F(E)$ **in low-energy limit**

W. Domcke, Atom. Mol. Phys. 16 359 (1983).

- attractive Coulomb

$$F(k) = \frac{A}{2\pi}\left[c - \frac{1}{2}ia_Bk + \log(-ia_Bk) + \psi\left(1 - \frac{i}{a_Bk}\right)\right]$$

- repulsive Coulomb

$$F(k) = -\frac{A}{2\pi}\left[c + \frac{1}{2}ia_Bk + \log(-ia_Bk) + \psi\left(1 + \frac{i}{a_Bk}\right)\right]$$

A : constant with dimension of energy

c : dimensionless constant

$\psi(x) = \frac{d}{dx}\log(\Gamma(x))$: digamma function

Coulomb+short range model

◉ short range limit $a_B \to \infty$

$$a_B = \frac{\hbar c}{\mu c^2 Z_1 Z_2}$$

$$-\frac{1}{a_s} + \frac{r_e}{2}k^2 - ik \pm \frac{2}{a_B}\left[\log(-ia_Bk) + \psi\left(1 + \frac{i}{a_Bk}\right)\right] = 0$$

$\to 0$

$$\longrightarrow -\frac{1}{a_s} + \frac{r_e}{2}k^2 - ik = 0$$ short range interaction

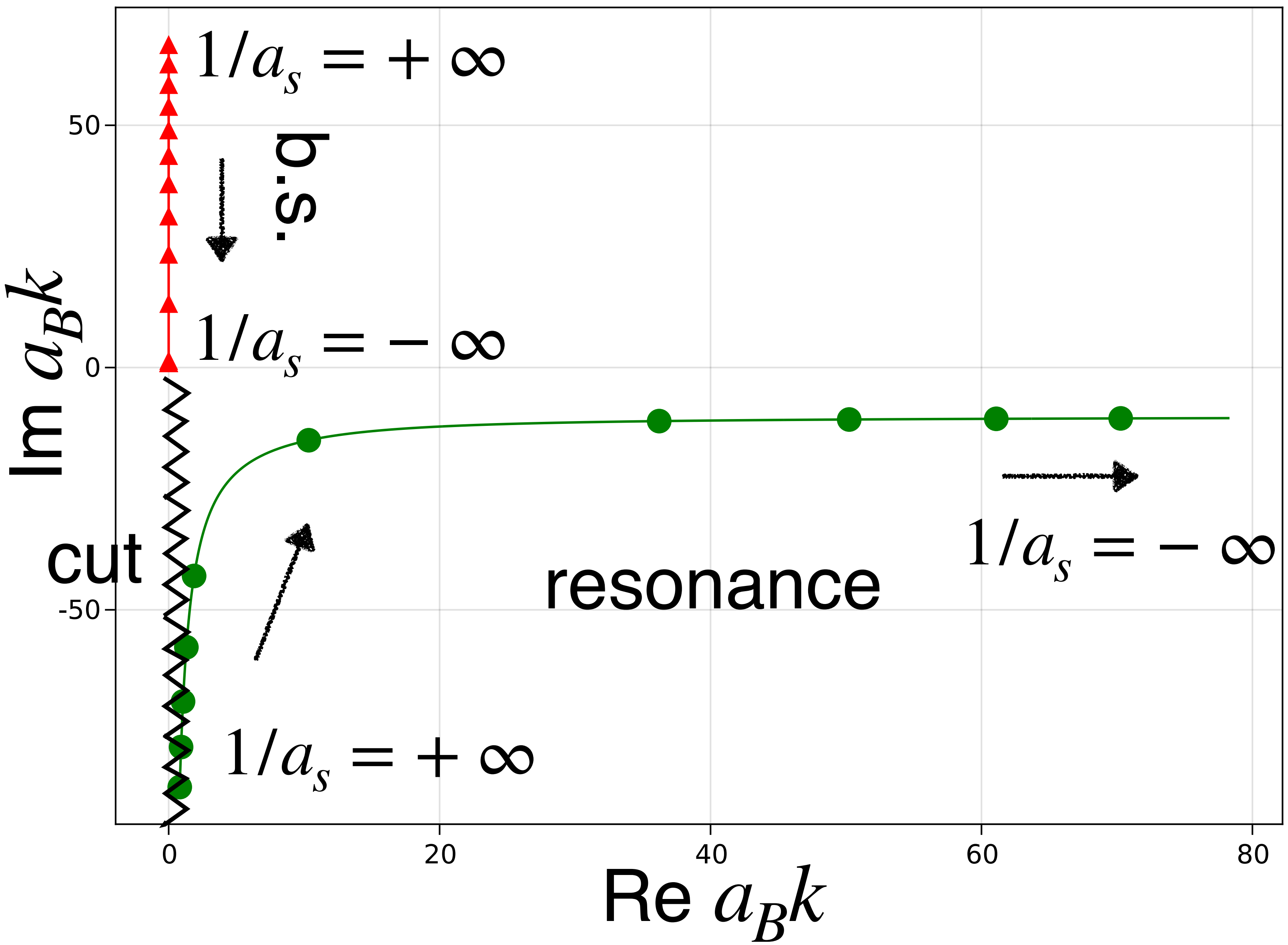
◉ further low-energy limit $r_e \to 0$

- zero-range theory S. Mochizuki, and Y. Nishida, Phys. Rev. C 110 , 064001 (2024).

$$\frac{ia_Bk}{2} \mp \log(-ia_Bk) + \psi\left(1 + \frac{i}{a_Bk}\right) + \frac{a_B}{2a_s} = 0$$

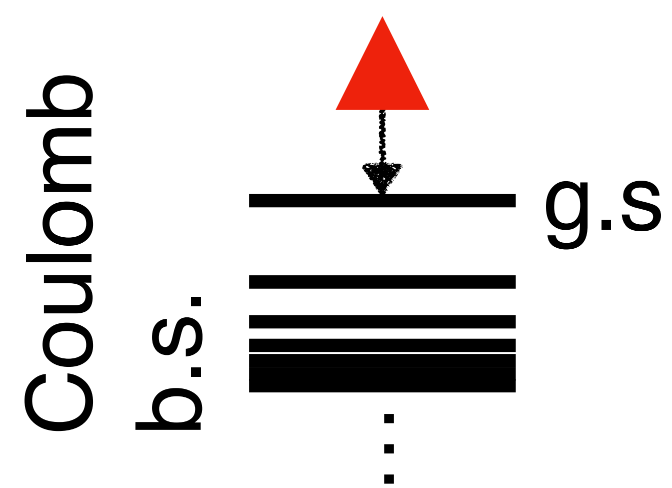
Pole trajectory (attractive Coulomb)

◉ pole trajectory in complex momentum k plane



- bound ≠ resonance

- bound pole
 → Coulomb g.s.

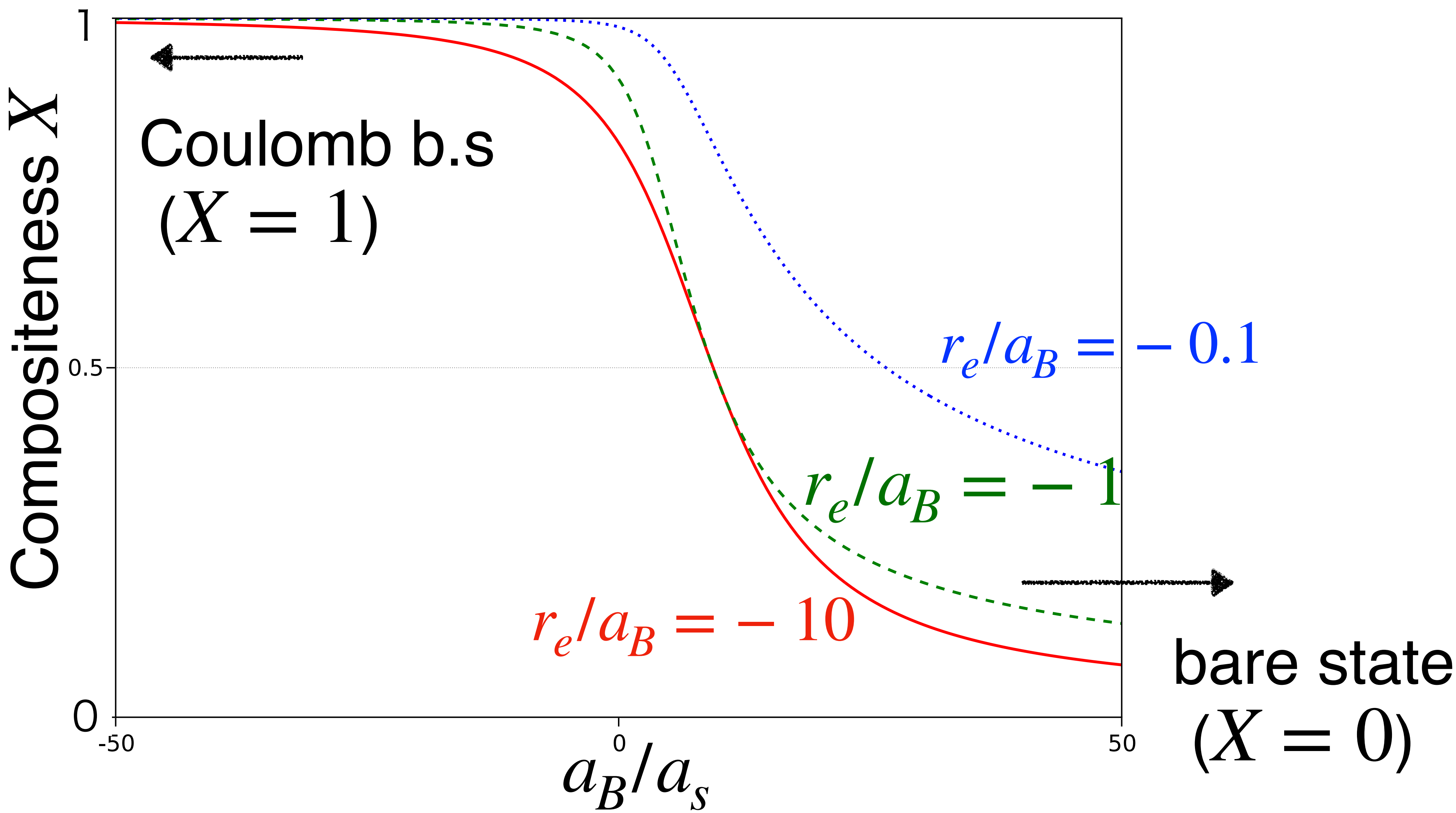


- pole cannot go to $k = 0$

W. Domcke, Atom. Mol. Phys. 16 359 (1983);

S. Mochizuki, and Y. Nishida,
Phys. Rev. C 110 , 064001 (2024).

Compositeness (att. Coulomb b.s.)



- $1/a_s \to +\infty$: states becomes elementary dominant ($X \to 0$)
- no short range universality but $X \to 1$ in $B \to B_{\text{Coulomb g.s.}}$ limit
  ∵ Coulomb g.s. has no bare state contribution (i.e. $X = 1$)
- Coulomb < short range ($r_e = -0.1$) : remnant of universality