The Sill distribution for p-wave and beyond

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Outline

In this presentation the following topics will be moved:

- Why is the Breit-Wigner distribution sometimes not sufficient?
- Introduction of the p-wave Sill distribution as a generalisation of the Sill distribution (for the s-wave);
- Application of this model to the decay of $\Delta(1232)^{++}$;

Introduction

In High-Energy Physics, the most popular energy distribution of the resonance comes in the form of the relativistic Breit-Wigner function:

$$d(s) = \frac{1}{\pi} \frac{M\Gamma}{(s - M^2)^2 + M^2\Gamma^2} \ . \tag{1}$$

Limitations: Lack of the lower-bound energy threshold. (Also, s cannot be negative.)

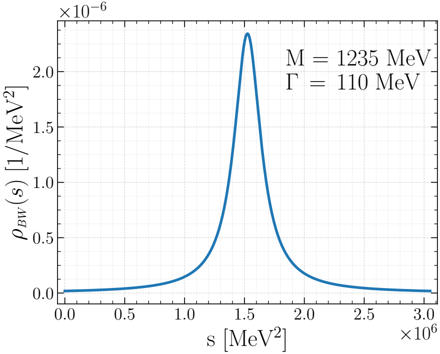
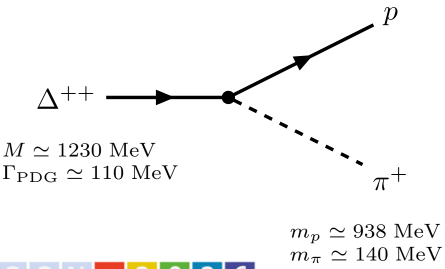


Figure 1: Plot of the Breit-Wigner distribution. Values taken for $\Delta(1232)^{++}$.

Introduction

The threshold problem becomes visible when the decay width Γ starts to be comparable with the difference between the mass M of the unstable state and the threshold energy E_{th} for its production.

This happens in the case of the $\Delta(1232)$ baryons family. In particular, the example of $\Delta(1232)^{++}$ will be studied later.



Conclusion: In such cases, the Breit-Wigner distribution should be replaced with different models that incorporate the proper threshold behaviour!

Non-relativistic Lee-Friedrich's model

In the Lee-Friedrich's model, the Hamiltonian operator H is composed of the **free part** H_0 and the **interaction part** H_1:

$$H = H_0 + H_1 \ . \tag{2}$$

The model assumes the existence of well-defined eigenstates of the free part H_0 corresponding to, respectively, the initial state $|S\rangle$ and the decay products $|\mathbf{k}\rangle$ with continuous spectra:

$$H_0 \, |S\rangle = M_0 \, |S\rangle \ , \quad H_0 \, |\mathbf{k}\rangle = \omega(\mathbf{k}) \, |\mathbf{k}\rangle \ . \tag{3}$$

H_1 introduces the interaction (mixing) between the in- and out-states and the full Hamiltonian takes the following form:

$$H = M_0 \, |S\rangle\langle S| + \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \omega(\mathbf{k}) |\mathbf{k}\rangle\langle\mathbf{k}| + \int \frac{\mathrm{d}^3 k}{(2\pi)^3} g(\mathbf{k}) (|\mathbf{k}\rangle\langle\mathbf{S}| + |\mathbf{S}\rangle\langle\mathbf{k}|) \ . \tag{4}$$

Lee-Friedrich's model – a tool to study resonances

The central object in the Lee-Friedrich's model is the **propagator**:

$$G(E) = \frac{1}{E - M_0 + \Pi(E) + i\varepsilon} \ , \tag{5}$$

which determines the time-evolution of the system with $\Pi(E)$ being the so-called **self-energy function** that incorporates the coupling between $|S\rangle$ and the continuum $|k\rangle$:

$$\Pi(E) = -\int_{E_{th}}^{\infty} \mathrm{d}E \frac{|g(\mathbf{k})|^2}{E - \omega(\mathbf{k}) + i\varepsilon} \ . \tag{6}$$

Moreover, the spectral function emerges from the propagator:

$$d_S(E) = -\frac{1}{\pi} \Im \left[G(E) \right] \ . \tag{7}$$

Relativistic Lee-Friedrich's model

The Lee-Friedrich's model may be extended to the relativistic energies (Eur. Phys. J. C (2020) 80:1191). The propagator $G(s)$ retains a similar form, but is expressed in terms of the squared invariant mass s instead of the energy E:

$$G(s) = \frac{1}{s - M_0^2 + \Pi(s) + i\varepsilon} \ , \tag{8}$$

with

$$\Pi(s) = -\int_{s_{th}}^{\infty} \mathrm{d}s' \frac{\rho_\Pi(s')}{s - s' + i\varepsilon} \ . \tag{9}$$

Also, similarly as before:

$$\rho(s) = -\frac{1}{\pi} \Im\left[G(s)\right] \ , \tag{10}$$

The Sill distribution (for the s-wave)

Instead of an explicit solution to the problem, one can try to model $\Pi(s)$. Such attempt has been performed e.g. by Giacosa et al. (Eur. Phys. J. A 57 (2021) 336). The introduced Sill distribution aims at the description of the energy threshold.

$$\Im\left[\Pi(s)\right] = \tilde{\Gamma}\sqrt{s - s_{th}} \propto k_{CM}\,, \quad \text{for } s \to s_{th}^{+}\,. \tag{11}$$

In this case, explicit calculations show that:

$$\Re\left[\Pi(s)\right] = 0\,, \tag{12}$$

which leads to the following formula describing the spectral function:

$$d(s) = \frac{\tilde{\Gamma}\sqrt{s - s_{th}}}{\pi} \frac{1}{(s - M_0^2)^2 + \tilde{\Gamma}^2(s - s_{th})}\,. \tag{13}$$

The Sill distribution – extension to any partial wave

The Sill distribution may be applied to the unstable systems with total orbital angular momentum equal to 0 ($l = 0$). **The main goal of this study is its extension to any other l.**

Once again, we start from the imaginary part of the self-energy $\Pi(s)$. For any l we propose:

$$\Im\left[\Pi(s)\right] = \frac{\tilde{\Gamma}(s - s_{th})^{l+1/2}}{s^l} \ , \tag{14}$$

with decay width $\Gamma(s)$ satisfying:

$$\Gamma(s) = \tilde{\Gamma}\left(\frac{s - s_{th}}{s}\right)^{3/2} \tag{15}$$

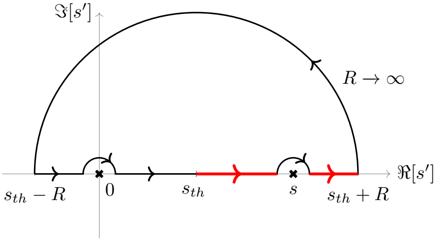
Note: The Sill distribution ($l = 0$) is the special case of the above formula.

Real part of the self-energy function

Important remark; In general, for $l \neq 0$, the real part of the self-energy function is no longer equal to zero!

The real part is connected to the imaginary part via Kramers-Krönig relations:

$$\Re\left[\Pi(s)\right] = \frac{1}{\pi} P \int_{s_{th}}^{\infty} \mathrm{d}s' \frac{\Im\left[\Pi(s')\right]}{s' - s}$$



Real-part of the self-energy function - final results

Finally, the evaluation of this integral leads to the formula for the real part of the self-energy function:

$$\Re\left[\Pi(s)\right] = \sum_{n=0}^{l-1} \tilde{\Gamma} \binom{l+\frac{1}{2}}{n} (-1)^{n-l} s_{th}^{l-n+1/2} \left[s^{n-l} - M_0^{2(n-l)} \right] \ . \quad (16)$$

In particular, for the special case of the p-wave $(l = 1)$ we get:

$$\Re\left[\Pi(s)\right] = -\tilde{\Gamma} \frac{s_{th}^{3/2}}{s} + \tilde{\Gamma} \frac{s_{th}^{3/2}}{M_0^2} \ . \quad (17)$$

Spectral function

To summarize this part, we come back to the spectral function and present the final expressions. In general:

$$\rho(s) = \frac{\Im\left[\Pi(s)\right]}{\pi} \frac{1}{\left(s - M_0^2 + \Re\left[\Pi(s)\right]\right)^2 + \left(\Im\left[\Pi(s)\right]\right)^2} \,. \tag{18}$$

In the particular case of the p-wave ($l = 1$), we reproduce the following distribution:

$$\rho(s) = \frac{1}{\pi} \frac{\frac{\tilde{\Gamma}(s-s_{th})^{3/2}}{s}}{\left(s - M_0^2 - \tilde{\Gamma}\frac{s_{th}^{3/2}}{s} + \tilde{\Gamma}\frac{s_{th}^{3/2}}{M_0^2}\right)^2 + \left(\frac{\tilde{\Gamma}(s-s_{th})^{3/2}}{s}\right)^2} \,. \tag{19}$$

Application to Δ baryons - cross section measurements

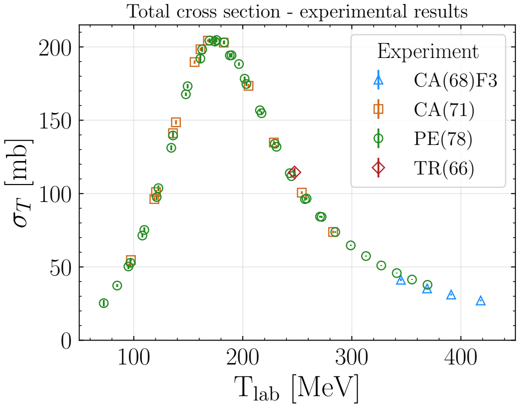


Figure 2: Total cross sections for $p - \pi^+$ scattering. (The Data Analysis Center at the George Washington University Institute for Nuclear Studies)

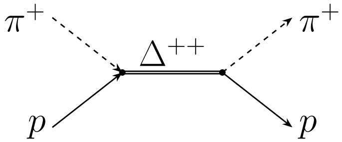
We employ our model to the data from the cross section measurements in pion-proton scattering.

The data comes from the following experiments:

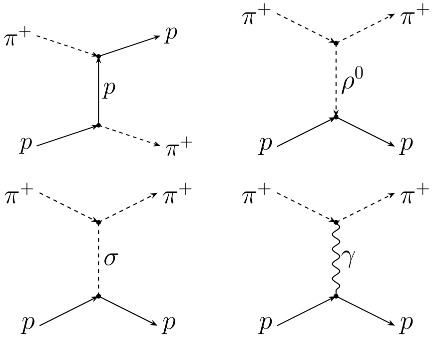
- CA(68)F3 – Carter et al., Phys. Rev. 168, 1457;
- CA(71) – Carter et al., Nuclear Physics B 26 (1971), 445-460;
- PE(78) – Pedroni et al., Nuclear Physics A, Vol. 300, No. 2, 1978, pp. 321-347;
- TR(66) – Troka et al., Phys. Rev. 144, 1115 (1966);

Model of the scattering amplitude

$\Delta(1232)^{++}$ formation - main process of interest



Additional (background) processes



... and more.

Model of the scattering amplitude

The scattering amplitude can be expressed as a sum of two components:

$$\mathcal{M}_{tot}(s) = \mathcal{M}_{\Delta}(s) + \mathcal{M}_{bg}(s)\ . \tag{20}$$

The term $\mathcal{M}_{\Delta}$ corresponds to the main process ($\Delta(1232)^{++}$ formation), while the term $\mathcal{M}_{bg}(s)$ includes all the other possible background processes:

$$\mathcal{M}_{bg}(s) = \frac{(s - s_{th})^{3/2}}{s} \sum_{n=0}^{\infty} c_n\,(s - s_{th})^n\ . \tag{21}$$

In the first order approximation, to maintain the proper threshold behaviour, we preliminarily model the background as follows (using also p-wave Sill):

$$\mathcal{M}_{bg}(s) = C\,\frac{(s - s_{th})^{3/2}}{s}\ \text{with}\ C \in \mathbb{R}\ . \tag{22}$$

Model of the scattering amplitude

Finally, the scattering amplitude can be written in the following form:

$$\mathcal{M}_{tot}(s) = \frac{\tilde{\Gamma}\frac{(s-s_{th})^{3/2}}{s}}{s - M_0^2 - \tilde{\Gamma}\frac{s_{th}^{3/2}}{s} + \tilde{\Gamma}\frac{s_{th}^{3/2}}{M_0^2} + i\tilde{\Gamma}\frac{(s-s_{th})^{3/2}}{s}} + C\,\frac{(s-s_{th})^{3/2}}{s}\;. \quad (23)$$

Moreover, the connection to the total cross section reads as follows:

$$\sigma_{tot}(s) = \frac{8\pi}{k^2(s)}\left|\mathcal{M}_{tot}(s)\right|^2\;. \quad (24)$$

Application to Δ baryons - Results

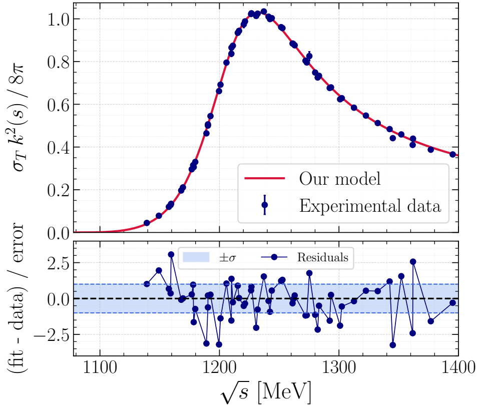


Figure 3: Results of the application of the model.

Fit results	
Parameter	Value
$\tilde{\Gamma}$	3099.7 MeV
M_0	1239.5 MeV
C	-0.000884 MeV^{-1}

$$\chi^2_\nu = 1.985\ , \quad \text{d.o.f.} = 55.$$

Position of the pole

$\sqrt{s_{pole}}$	Fit	PDG
$\Re\left[\sqrt{s_{pole}}\right]$	1210.0 MeV	1212.5 MeV
$-2\Im\left[\sqrt{s_{pole}}\right]$	100.9 MeV	97.4 MeV

Application to △ baryons - Results

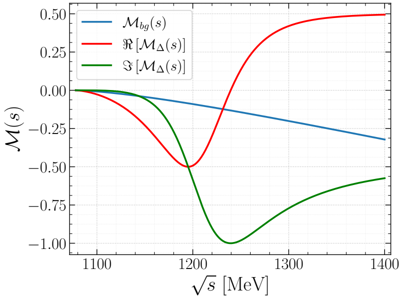


Figure 4: The contribution of different components to the scattering amplitude.

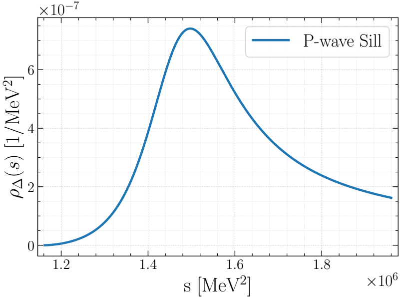


Figure 5: Plot of the spectral function (for the main process).

Application to Δ baryons - Results

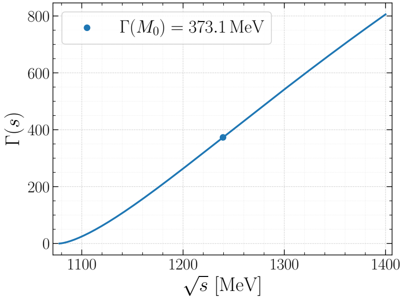


Figure 6: Decay width $\Gamma(s)$ as a function of energy.

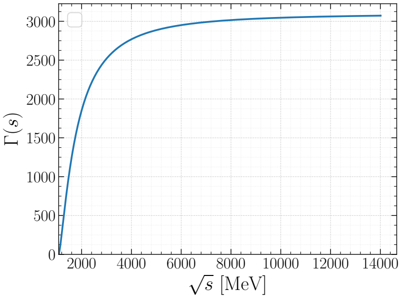


Figure 7: Decay width $\Gamma(s)$ as a function of energy.

Thank you for the attention!

Title: The Sill distribution for p-wave and beyond

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Backup - Breit-Wigner

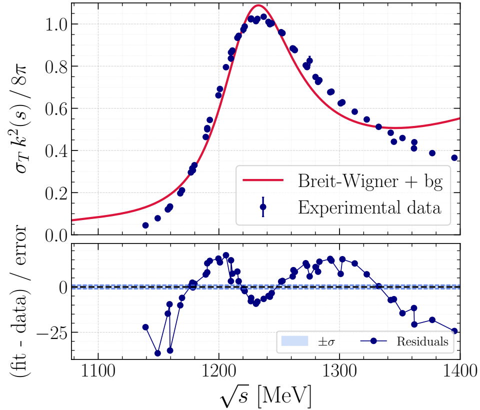

Figure 8: Results of the application of the model.

Fit results	
Parameter	Value
Γ	74.6 MeV
M_0	1224.0 MeV
C	0.00152 MeV^{-1}

Position of the pole		
$\sqrt{s_{pole}}$	Fit	PDG
$\Re\left[\sqrt{s_{pole}}\right]$	1224.6 MeV	1212.5 MeV
$-2\Im\left[\sqrt{s_{pole}}\right]$	74.5 MeV	97.4 MeV

Backup - Sill distribution

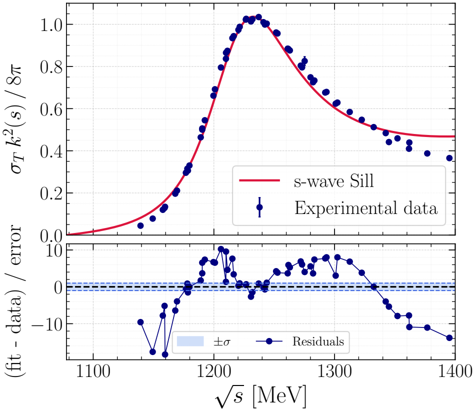

Figure 9: Results of the application of the model.

Fit results	
Parameter	Value
$\tilde{\Gamma}$	188.4 MeV
M_0	1225.8 MeV
C	0.000961 MeV^{-1}

Position of the pole		
$\sqrt{s_{pole}}$	Fit	PDG
$\Re\left[\sqrt{s_{pole}}\right]$	1219.4 MeV	1212.5 MeV
$-2\Im\left[\sqrt{s_{pole}}\right]$	99.0 MeV	97.4 MeV