EXCITED HADRONS FROM LATTICE QCD EFFECTIVE FIELD THEORIES PHENOMENOLOGY

MESON2026 — June 2026 — KRAKÓW Auditorium Maximum

MAXIM MAI

UNIVERSITY OF BERN

THE GEORGE WASHINGTON UNIVERSITY

DFG: Heisenberg Programme (532635001)
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DFG Project-ID 196253076
DOE: DE-SC0016582, DE-AC05-06OR23177, DE-FG02-95ER40907
NSF: PHY-2012289

HADRON SPECTRUM

Experimental progress

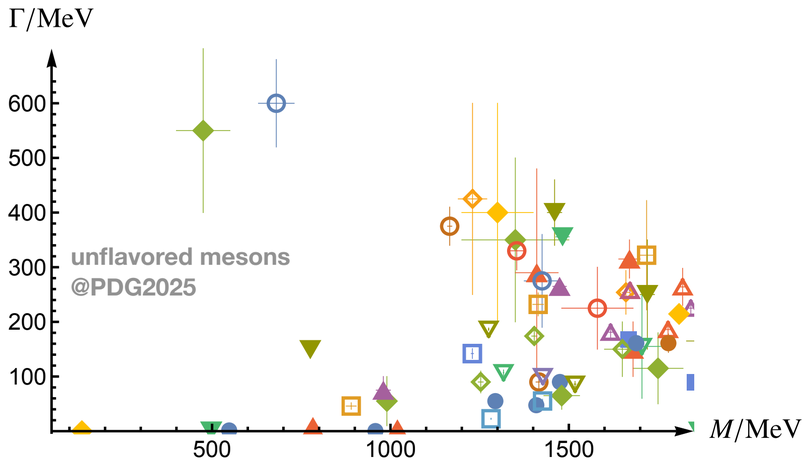
- 70y research ($\Delta(1232), \rho(770), \omega(782), \ldots$)
- mostly excited states

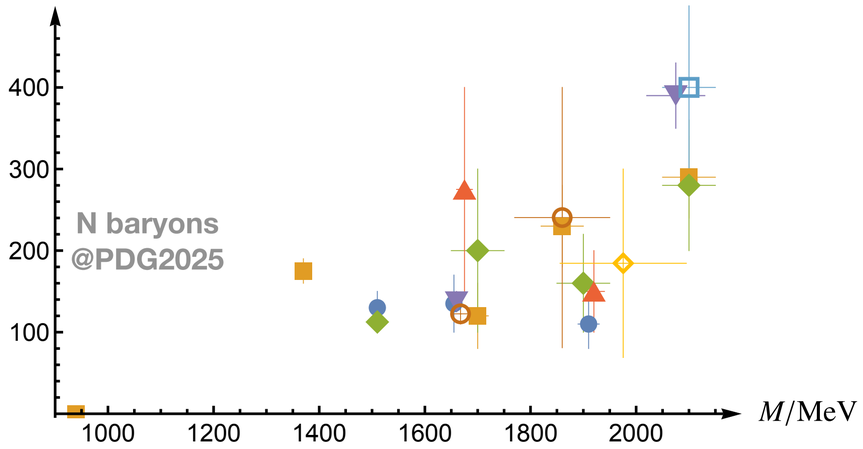
  ≈ 100 mesons + 50 baryons (****)
- ongoing experiments @CERN, GSI, JLAB, BES…

Talks: Nilsson, Rossi, Gradl, Sgaramella, …

New techniques

- Lattice QCD
  - Effective Field Theories
    - Machine Learning
      - Amplitude analyses …

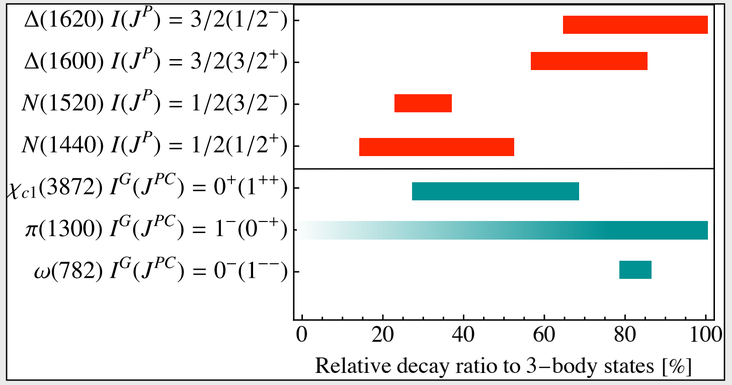




Review: MM/Meißner/Urbach Phys.Rept. 1001 (2023) 1-6

COMPLICATIONS

- overlapping states
- connection to QCD
- many states have large **three-body content**



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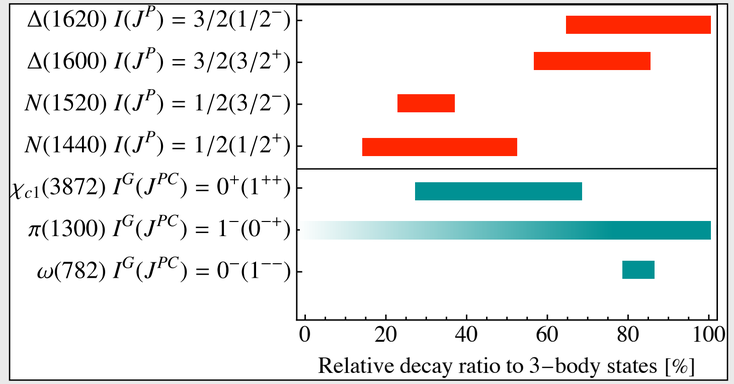


Consistency frontier

- Is excited pion so heavy?
- Why is excited nucleon so light?

…

Talks:
Ciepał, Hartmann, …

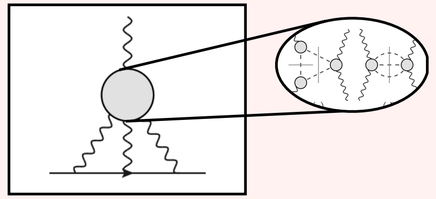
Discovery frontier

- Spin-exotic states
- Left-hand-cuts: $T_{cc}(3875)$
- Triangle singularities: $a_1(1420)$

…

Talks:
Dawid, Molina, Sakthivasan, Belov, Zhang, Winney, Yi, Hanhart, Guo…

Precision frontier



- Hadronic light-by-light
- τ-EDM
- Neutron stars

…

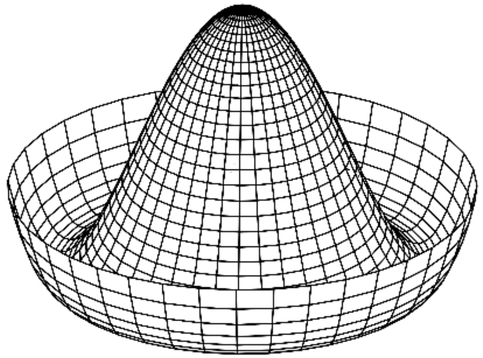
Talks:
Danilkin, Bondar, Del Grande, Roberts, …

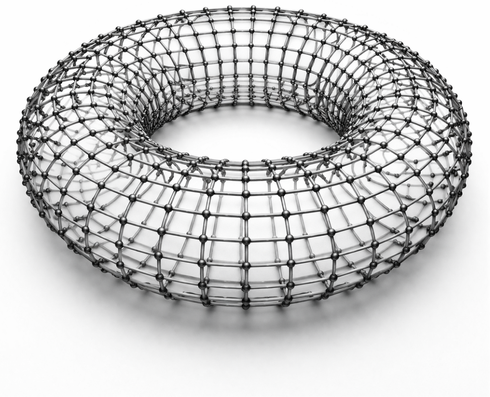
THEORY TOOLS

EFFECTIVE FIELD THEORIES

LATTICE QCD

...



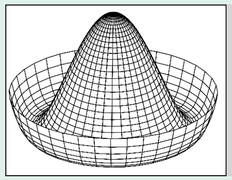


NON-PERTURBATIVE APPROACHES

Effective Field Theory (EFT/CHPT)

Weinberg (1979) Gasser, Leutwyler (1981), …

Reviews: Ann. Rev. Nucl. Part. Sci. 57, 33 (2007), Adv. Nucl. Phys. 27, 277 (2003), …



$$Z[J] = \int [DU] e^{\int id^4x \, \mathscr{L}_{\text{eff}}(U,v,a,s,p)}$$

[+] effective/hadronic degrees of freedom

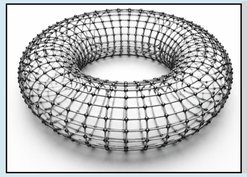
[-] unknown low-energy constants

[+] well-defined power counting/quark mass dependence

Lattice Gauge Theory (LQCD)

Wilson, Phys. Rev. D10 (1974) 2445 , …

Reviews: hep-lat/9807028 [hep-lat] Rev.Mod.Phys. 90 (2018) Rept.Prog.Phys. 86 (2023)



$$Z[J] = \int [DU] e^{-S_E} \det[M[U]]$$

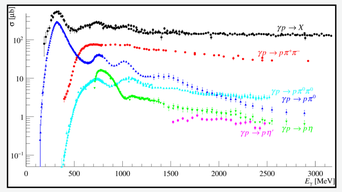
[+] non-perturbative access to QCD Green's functions

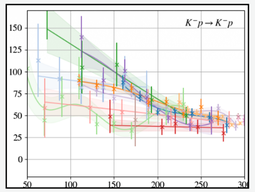
[-] discretized space-time / Euclidean metric / finite-volume

[+/-] unphysical quark masses

S-matrix

Crossing symmetry — particle/antiparticle
Unitarity — probability conservation
Analyticity — causality





Further approaches: Functional methods, holography, K-matrix, dynamical models, …

Review: Eichmann/Sanchis-Alepuz/Alkofer/Fischer Prog.Part.Nucl.Phys. 91 (2016) 1-100

Review: MM/Meißner/Urbach Phys.Rept. 1001 (2023) 1-6

Review: Döring/Haidenbauer/Sato/MM PPNP(2025)

EXAMPLE: 2-BODY CASE $\Lambda(1405), \Lambda(1380), \Lambda(1670)$

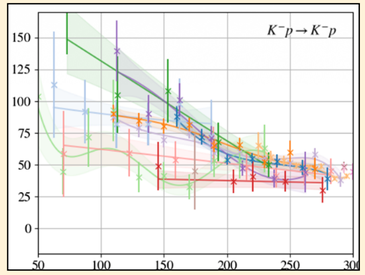
Talks: Cieply, …

INPUT #1: Physical point

- cross sections, SIDDHARTA(2) **Talk: *Sgaramella***
- Amplitude

$$T_l = \frac{1}{K^{-1} - i p_{\rm cm}}$$

- $Im\ T_l^{-1}$ **on-shell configurations**



INPUT #2: Lattice QCD (unphysical point)

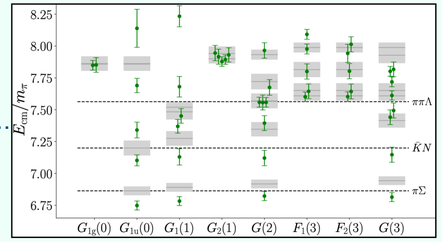
[BaSc] PRL 132 (2024) 5; Sucunza+ arXiv:2605.28447

- discrete spectrum
- 2b Quantization Condition

Lüscher, NPB 354, 531 (1991), …

$$\frac{1}{K^{-1}(E^*) - B_L(E^*)} = \infty$$

- $B_L = \infty$ **on-shell configurations**



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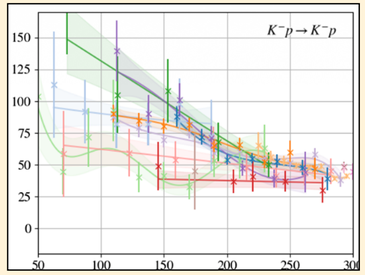

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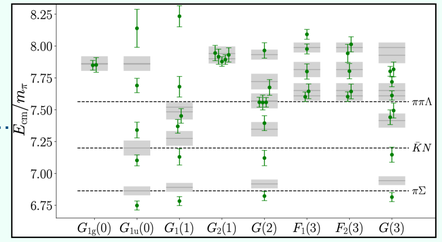
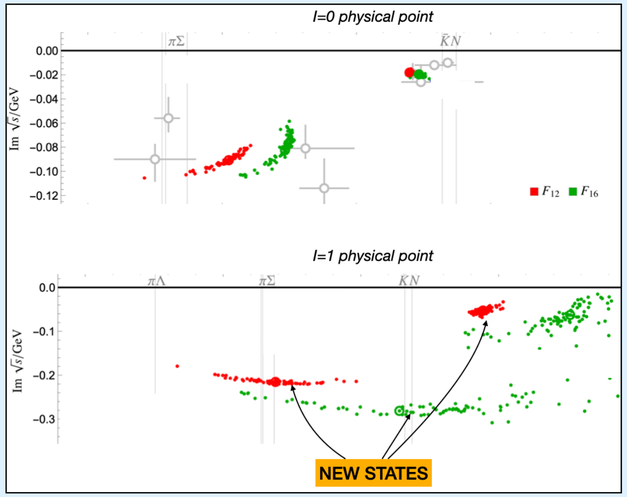


OUTPUT

- Unknown quantity K^{-1}
- Quark-mass dependence from CHPT

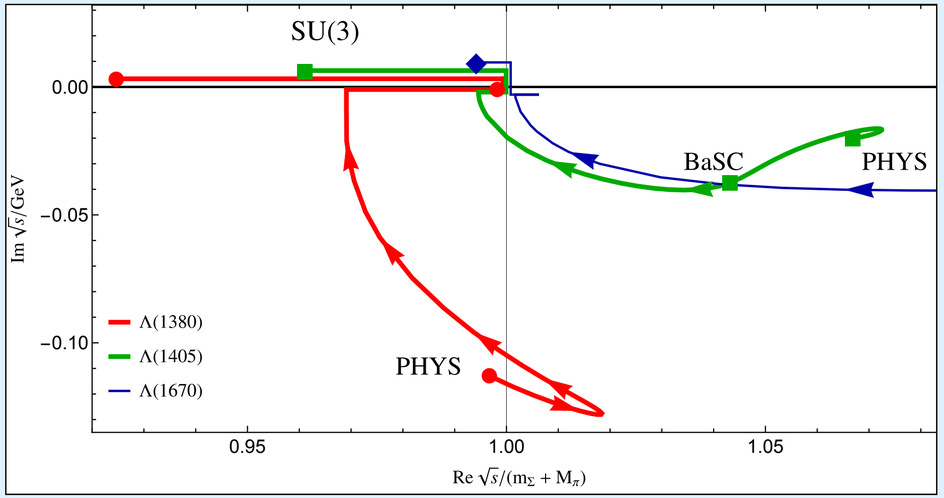
$$\mathcal{L}^{(2)}_{\phi B} = b_{D/F}\langle\bar{B}[\chi_+, B]_\pm\rangle + b_0\langle\bar{B}B\rangle\langle\chi_+\rangle + b_{1/2}\langle\bar{B}[u_\mu, [u^\mu, B]_\mp]\rangle + \\ + i\sigma^{\mu\nu}(b_{5/6}\langle\bar{B}[[u_\mu, u_\nu], B]_\mp\rangle + b_7\langle\bar{B}u_\mu\rangle\langle u_\nu B\rangle) + \frac{ib_{8/9}}{2m_0}(\langle\bar{B} \\ + \frac{ib_{10}}{2m_0}(\langle\bar{B}\gamma^\mu\{u_\mu, \{u_\nu, [D^\nu, B]\}\}\rangle + \langle\bar{B}\gamma^\mu[D_\nu, \{u^\nu, \{u_\mu, B\}\}]\rangle \\ + \langle\bar{B}\gamma^\mu B\rangle\langle[D_\nu, u_\mu]u^\nu + u_\mu[D_\nu, u^\nu]\rangle),$$

pole positions



Pittler+ Phys.Rev.D 112 (2025) 7, 074037

chiral trajectory



Guo+ *Phys.Lett.B* 846 (2023) 138264 Sucunza+ arXiv:2605.28447

3-BODY CASE: FVU APPROACH

Three-body scattering amplitude

- 3b-Unitarity
- intergal equation (explicit momenta)
- generalized to all channels + strangeness

Feng+ PRD 110 (2024) 094002 2407.08721

"Infinite Volume Unitarity" (IVU)

$$T(s,p,p') = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B+C)}{2E_l} \frac{1}{\tilde{K}^{-1} - \Sigma} T$$

MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)

on-shell configurations
no free parameters

Real/offshell
"2-/3-body force"

Related: Poleyaeva (2012), Hansen+ (2014), Hammer+ JHEP 10 (2017) 115, Wunderlich+ JHEP 08 (2019); Jackura+ Eur.Phys.J.C 79 (2019); Kang Yu+ 2603.07205 [hep-lat], Dawid+ Phys.Lett.B 864 (2025) 139442…
Reviews: Hansen/Sharpe: Ann.Rev.Nucl.Part.Sci. 69 (2019) 65-107, MM/Rusetsky/Doring *Eur.Phys.J.ST* 230 (2021) 6, 1623-1643

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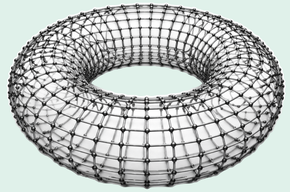
on-shell configurations
no free parameters

Real/offshell
"2-/3-body force"

Three-body quantization condition

- only discrete momenta are possible $2\pi/L\,\mathbb{Z}^3$
- determinant equation
- relevant volume-dependence: singularities/onshell states $(+O(e^{-ML}))$



$\sim O((ML)^n)$ $\sim O(e^{-ML})$

"Finite Volume Unitarity" (FVU)

$$\det\left[\, 2L^3 E_{\mathbf{p}} \left(\tilde{K}^{-1} - \Sigma^L\right) - B - C \,\right]^{\Gamma} \equiv 0$$

MM/Döring Eur.Phys.J.A 53 (2017) 12, 240

Related: Poleyaeva (2012), Hansen+ (2014), Hammer+ JHEP 10 (2017) 115, Wunderlich+ JHEP 08 (2019); Jackura+ Eur.Phys.J.C 79 (2019); Kang Yu+ 2603.07205 [hep-lat], Dawid+ Phys.Lett.B 864 (2025) 139442…
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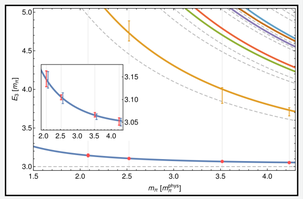
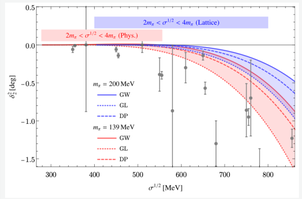
$\pi^+\pi^+\pi^+$

Phys.Rev.Lett. 122 (2019)



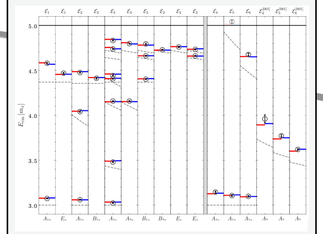
$\pi^+\pi^+\pi^+$

Phys.Rev.D 101 (2020) 5



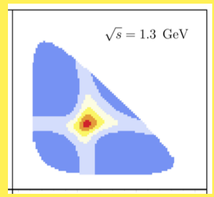
$\pi^+\pi^+\pi^+$

Phys.Rev.D 101 (2020) 11



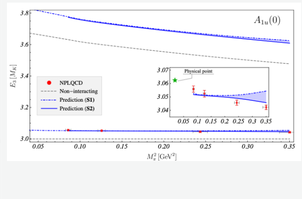
$a_1(1260)$

Phys.Rev.D 101 (2020) 9



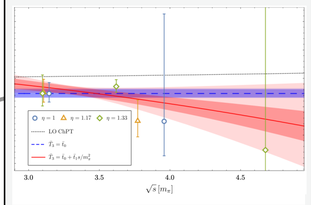
$K^-K^-K^-$

Phys.Rev.D 102 (2020) 11



$\pi^+\pi^+\pi^+$

Phys.Rev.D 104 (2021)



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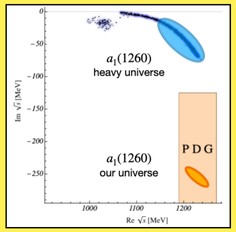
Phys.Rev.Lett. 127 (2021)

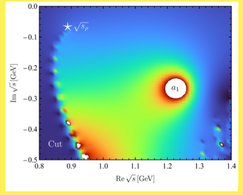


Eur.Phys.J.A 53 (2017) 12, 240

FVU/IVU APPLICATIONS

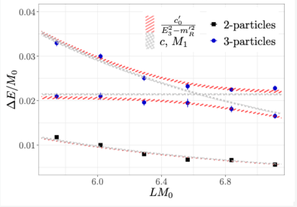
$a_1(1260)$

Phys.Rev.D 105 (2022)



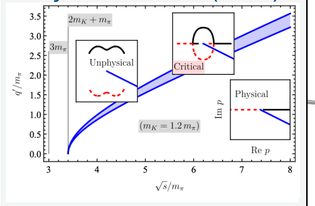
3ϕ

JHEP 02 (2023) 252



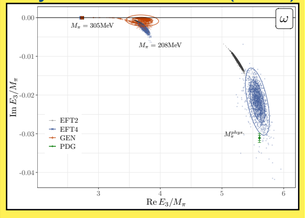
$3 \times (\pi/K)$

Phys.Rev.D 110 (2024)



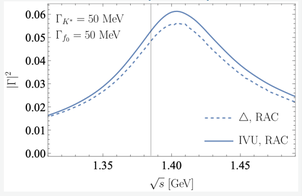
$\omega(782)$

Phys.Rev.Lett. 133 (2024)



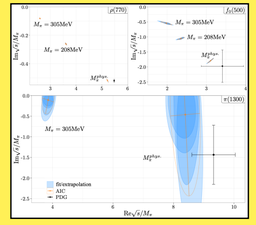
Triangle sing.

JHEP 10 (2024) 246



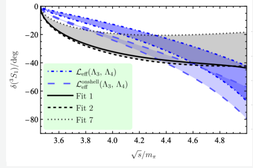
$\pi(1300)$

2510.09476 in press PRL



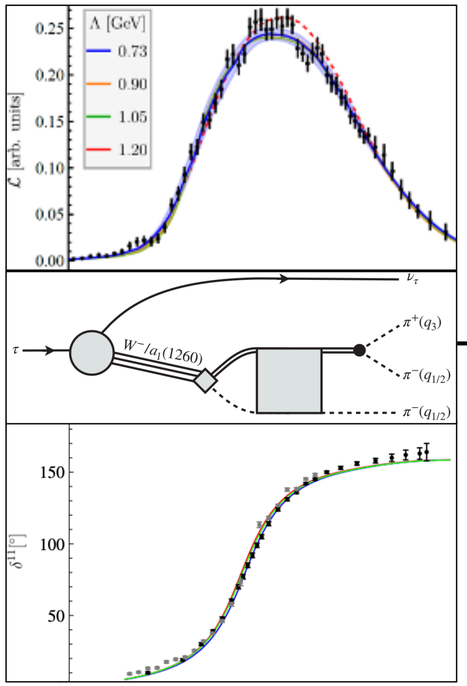
Isotensor $\pi\pi\pi$

2601.16916



AXIAL-VECTOR MESON

Sadasivan/Akdag/MM/Culver/Alexandru/Lee/Doring Phys.Rev.D 101 (2020); *Phys.Rev.D* 105 (2022)



Schael et al. [ALEPH] Phys.Rept. 421 (2005); Nucl.Phys.B 79; Phys.Rev.D 7;

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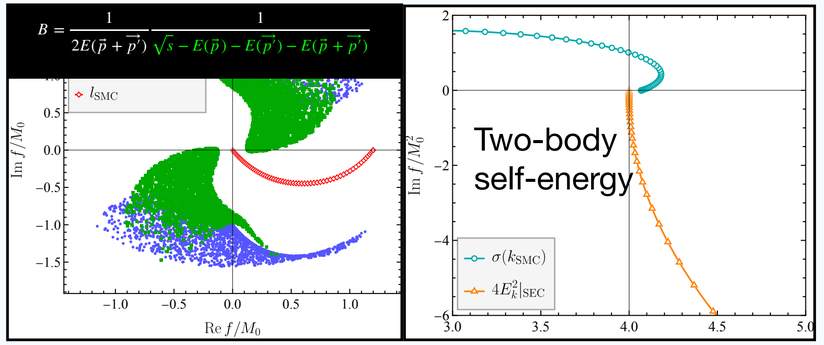
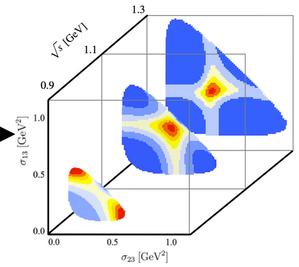
MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)

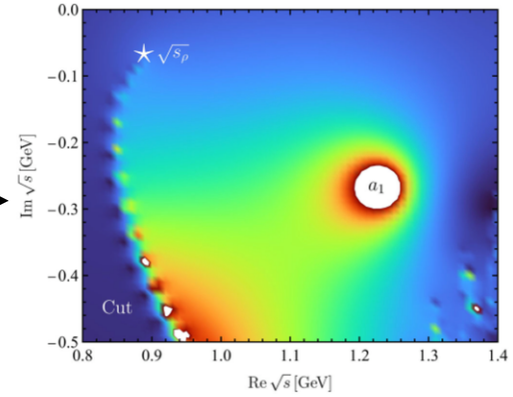
Solution techniques

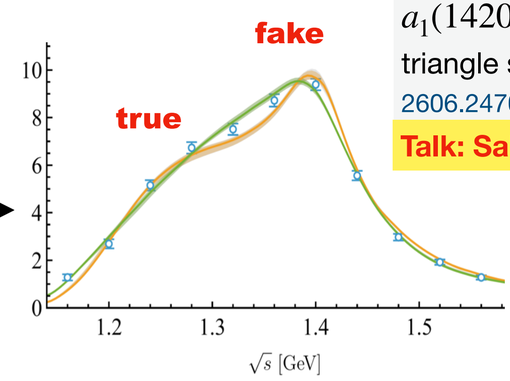
Hetherington/Schick, Phys. Rev. 137, B935–B948 (1965). Cahill/Sloan, Nucl. Phys. A 165, 161–179 (1971) Schmid/Ziegelmann, Adhikari/Amado, Phys. Rev. D 9, 1467–1475 (1974)
Comparison: Sakthivasan et al. *JHEP* 10 (2024) 246

- Complex contour deformation $|\ell| \in \mathbb{C}$
- Analytic continuation to the real axis









$a_1(1420)$ as triangle singularity
2606.24709
Talk: Sakthivasan

Review on triangle singularities: Guo+ Prog.Part.Nucl.Phys. 112 (2020) 103757

AXIAL-VECTOR MESON (LATTICE QCD)

MM/Culver/Alexandru/Doring/Lee/Sadasivan [GWQCD] PRL 127 (2021)

IVU

$$T(s,p,p') = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B+C)}{2E_l} \frac{1}{\tilde{K}^{-1} - \Sigma} T$$

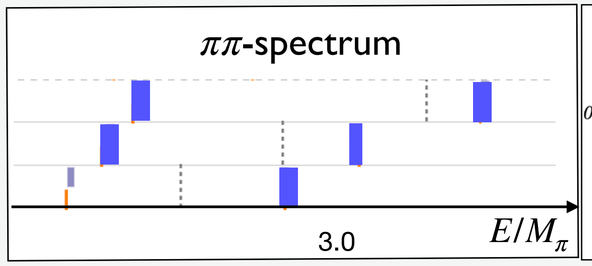
$C, \tilde{K}$

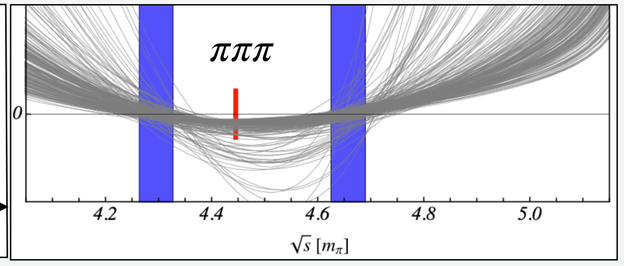
- volume-independent
- generic form

FVU

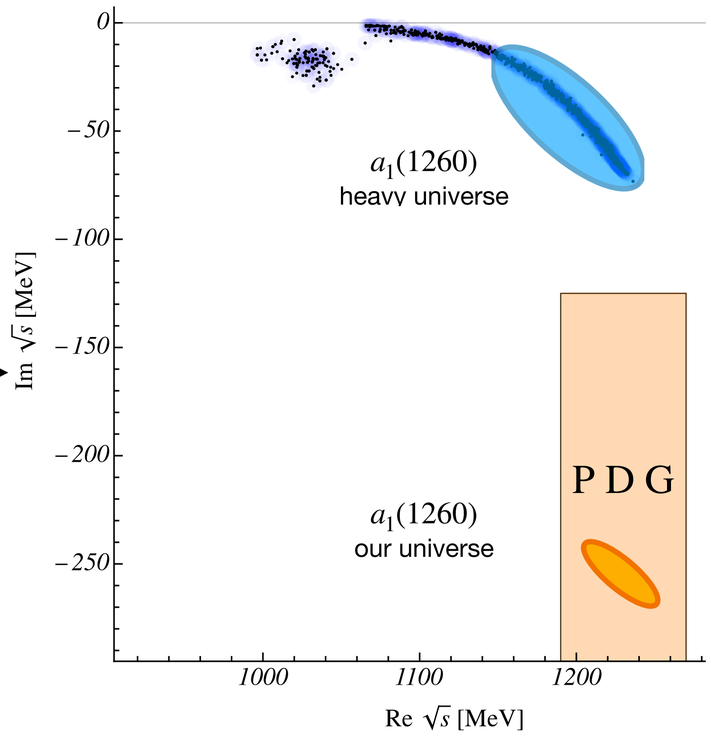
$$\det\left[\, 2L^3 E_{\mathbf{p}} \left(\tilde{K}^{-1} - \Sigma^L\right) - B - C \,\right]^{\Gamma} \equiv 0$$

Lattice QCD ($M_\pi \approx 315\text{MeV}, N_f = 2$)





[GWQCD] PRD94(2016) PRD98 (2018) PRD 100(2019)

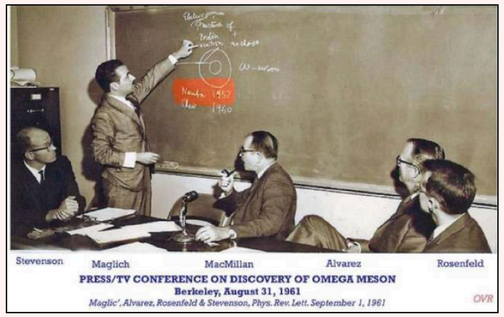


VECTOR MESON

[agemo] Yan+ PRL133 (2024)

$\omega(782)$

- lightest hadron decaying into three particles



- isoscalar response within the VMD of the photon-nucleon interactions ***Talks: Roberts***
- ~repulsion at < 1 fm in the one-boson-exchange picture of the N–N interaction

Sakurai (1960); Erkelenz (1974); Brown and Jackson (1976); Barkov et al., 1985; Connell et al. (1997); Bazavov et al. (2021)

What can we learn from Lattice QCD?

- two/three-body force
- pion-mass dependence
- KSFR/Universality relations/… in EFT

Gell-Mann/Sharp/Wagner/Fujiwara/Kawarabayashi… Review: Meißner Phys.Rept. 161 (1988) 213

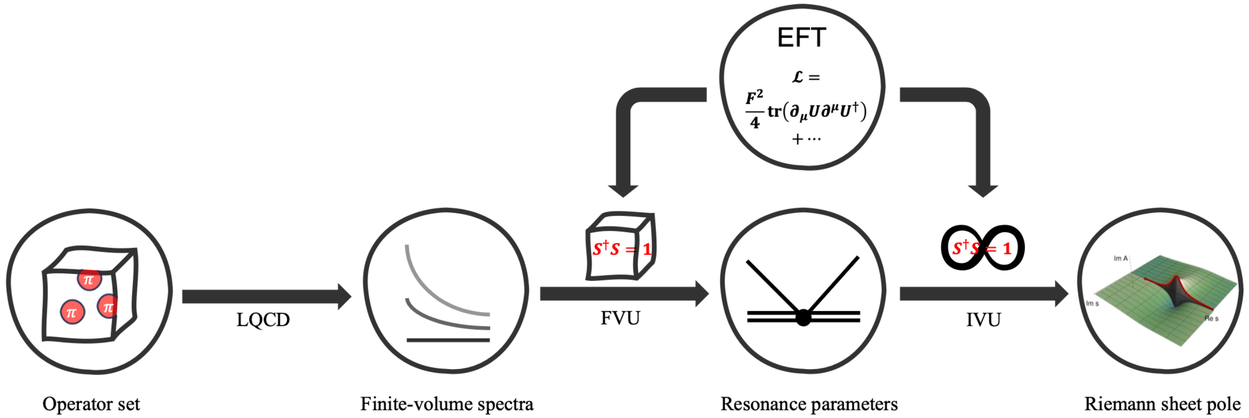


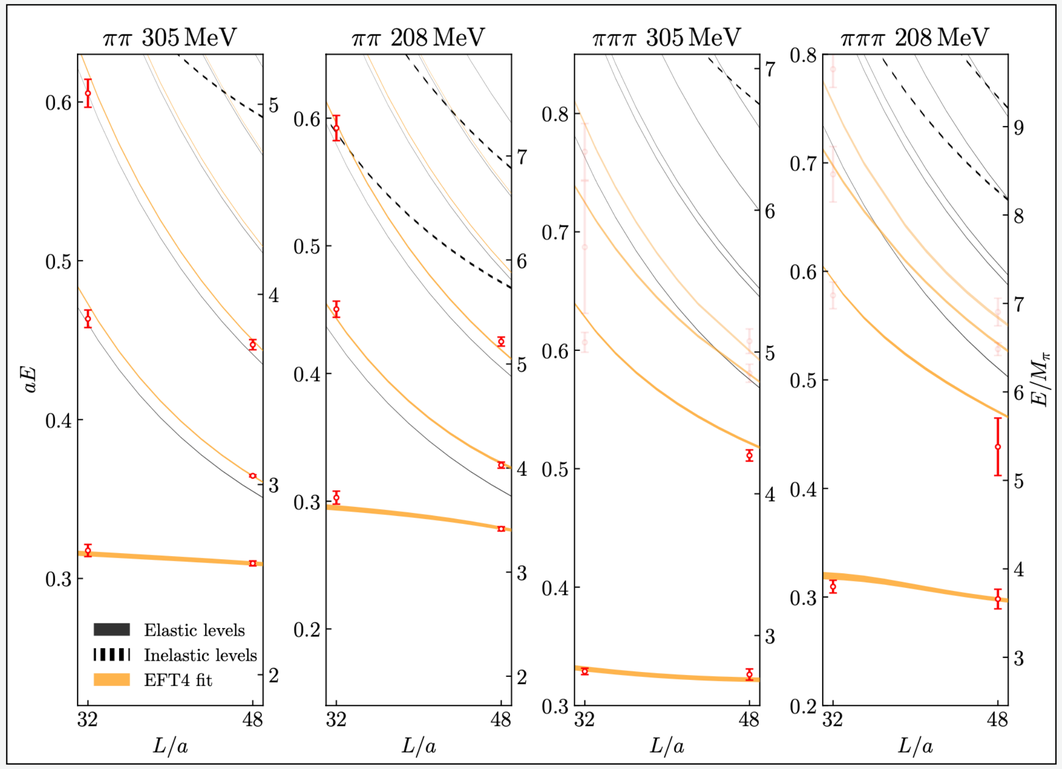
Fig by Haobo Yan (燕浩波)

VECTOR MESON

[agemo] Yan+ PRL133 (2024)

Lattice QCD input

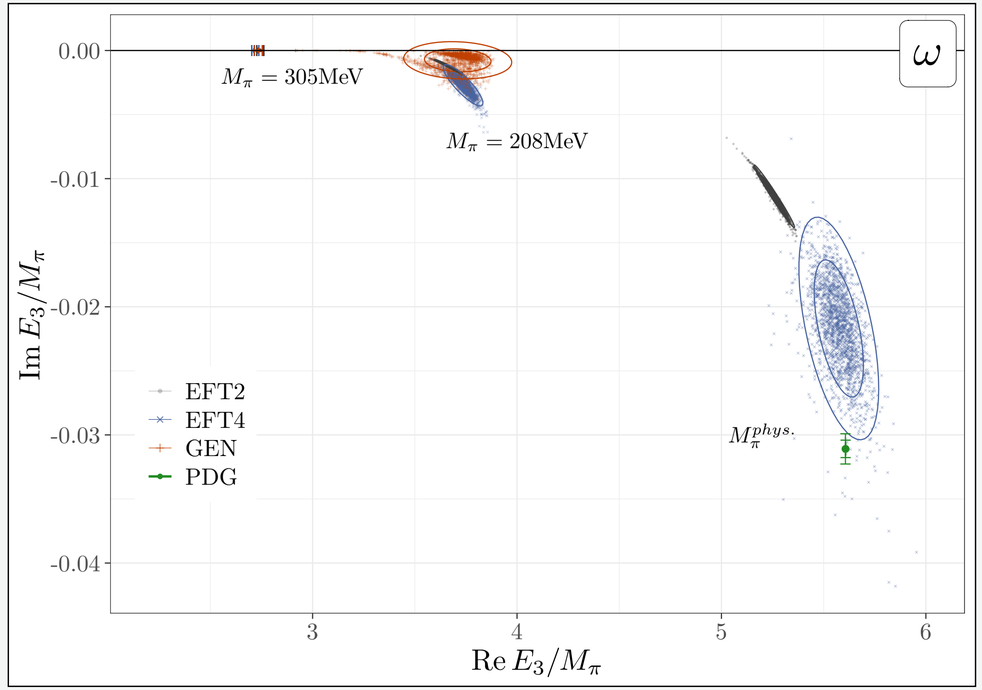
- Nf = 2 + 1 Clover fermions
- 2/3 particle operators
- 2 pion masses ($\approx 210, 305$ MeV) 2 volumes ($L^3 = 32^3, 48^3$)



FVU

$$\det\left[\, 2L^3 E_{\mathbf{p}} \left(\tilde{K}^{-1} - \Sigma^L\right) - B - C \,\right]^{\Gamma} \equiv 0$$

- Various EFT based ansatzes
- $\omega(782)$ becomes abound state at ~300 MeV
- at the physical point very close to the EXP value



EXCITED PION

[agemo] Yan+ PRL136 (2026) no.14, 141901

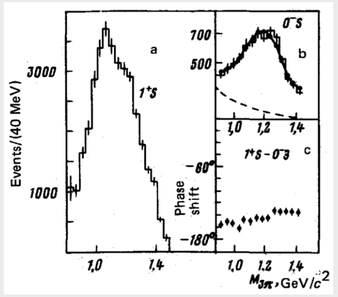
$\pi(1300)$ $I^G(J^{PC}) = 1^-(0^{-+})$

$\pi(1300)$ MASS	[1]	1300 ± 100 MeV
$\pi(1300)$ WIDTH	[1]	200 to 600 MeV

$\pi(1300)$ **DECAY MODES**

Mode		Fraction (Γ_i / Γ)	Scale Factor/ Conf. Level	P(MeV/c)
Γ_1	$\rho\pi$	seen		404
Γ_2	$\pi(\pi\pi)_{S-\text{wave}}$	seen		

Discovery: $\pi Z \to \pi\pi\pi Z$



Bellini+ [Bologna-Dubna-Milan-Collaboration] JETP Lett. 34 (1981) 488

Most recent analysis $D \to 4\pi$

- Strong correlations between PWA contributions

	$m_{a_1(1260)}$	$\Gamma_{a_1(1260)}$	$m_{a_1(1640)}$	$\Gamma_{a_1(1640)}$	$m_{\pi(1300)}$	$\Gamma_{\pi(1300)}$
$m_{a_1(1260)}$	+1.000	+0.689	−0.065	−0.282	+0.116	−0.258
$\Gamma_{a_1(1260)}$		+1.000	−0.114	−0.176	+0.013	−0.004
$m_{a_1(1640)}$			+1.000	−0.335	−0.136	−0.119
$\Gamma_{a_1(1640)}$				+1.000	−0.258	+0.370
$m_{\pi(1300)}$					+1.000	−0.425
$\Gamma_{\pi(1300)}$						+1.000

- Existence required by data?

  d'Argent+ JHEP 05 (2017) 143

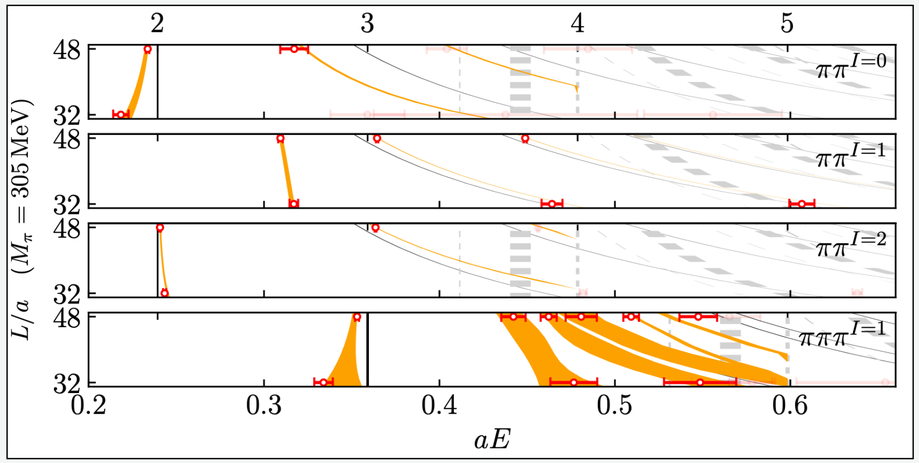
Phenomenology

- Very heavy: $M_{\pi(1300)} \approx 10 M_\pi \approx M_{N(1440)}$
- Phenomenologically hard to access:

  overlapping effects $a_1(1260), \ldots$
- Width very uncertain

$\pi(1300)$ FROM LATTICE QCD

[agemo] Yan+ PRL136 (2026) no.14, 141901



Lattice QCD input

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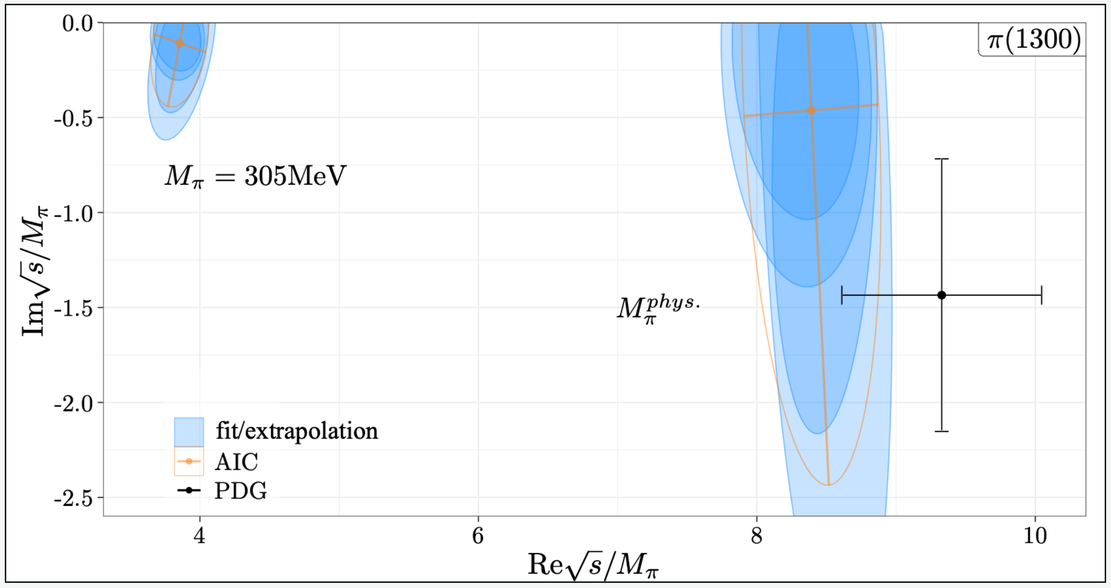
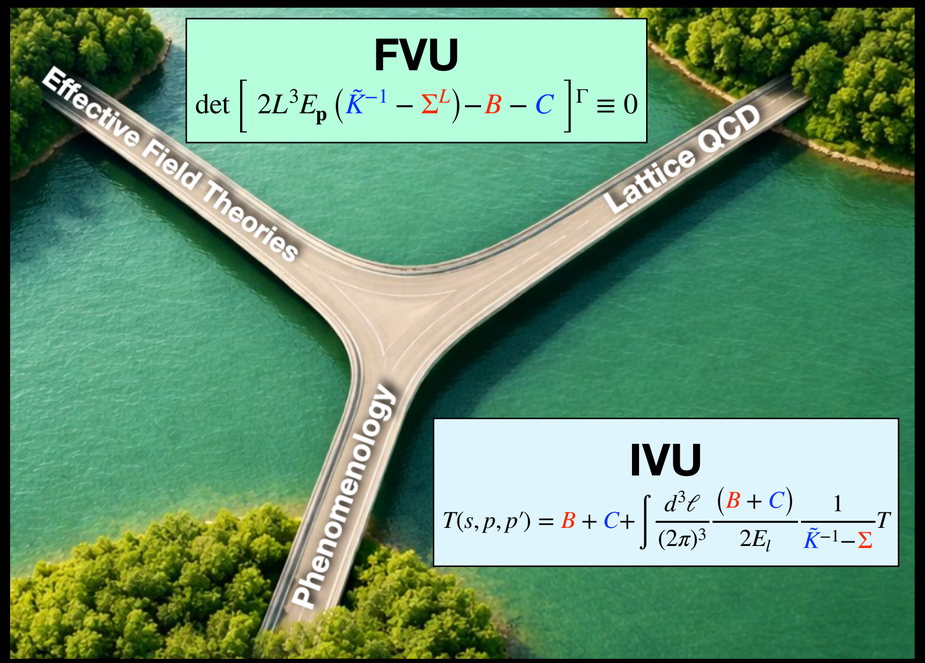
- 2/3 particle operators
- 2 pion masses ($\approx 210, 305$ MeV) 2 volumes ($L^3 = 32^3, 48^3$)
- Contractions (~400)



Excited state of the pion

- $M_\pi = 305\,\mathrm{MeV}$ finite-volume ensemble
  - ~~**no-resonance scenario**~~
  - Includes *dynamically generated* scenarios
- Physical point extrapolation
  - Mass/Width agree with PDG (1σ)

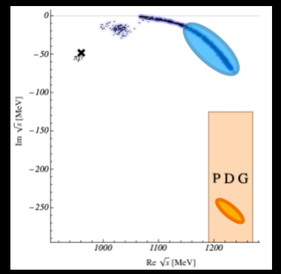


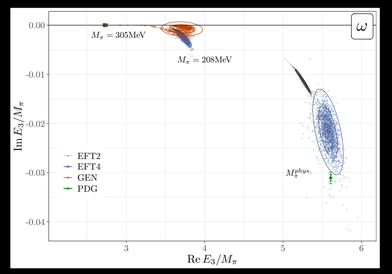
New discoveries from QCD

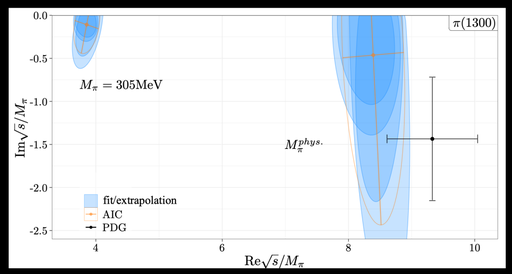
- Ab-initio Lattice input
- quark-mass extrapolations through EFT

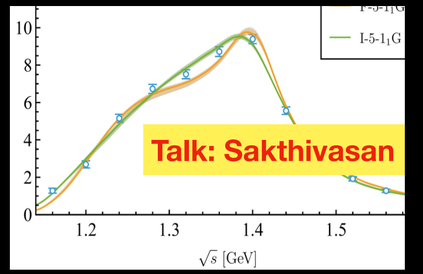
S-matrix & 3-body unitarity

- On/off-shell configurations
- **new 3-body quantization condition**

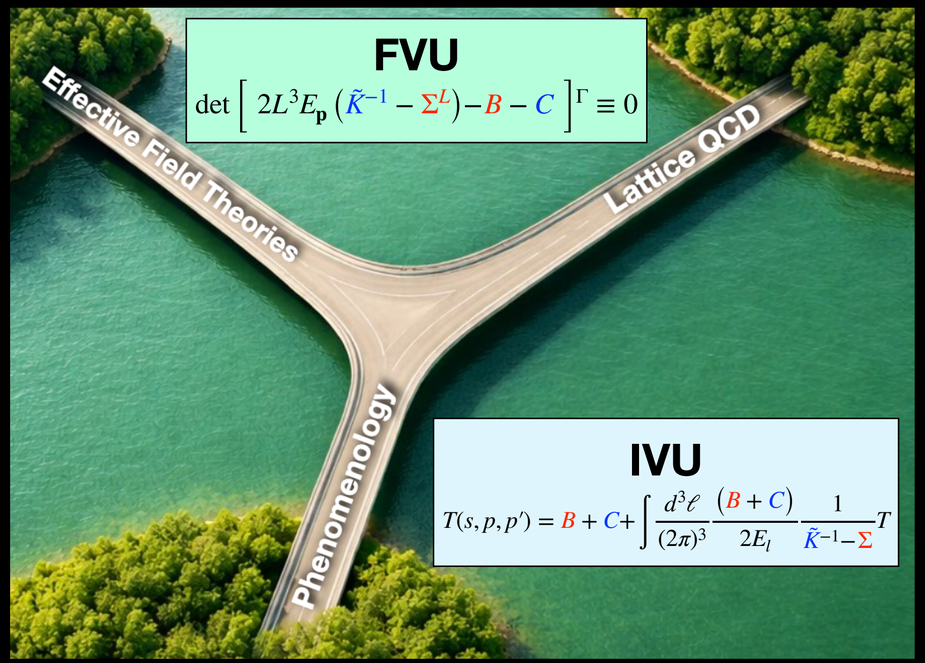








SUMMARY



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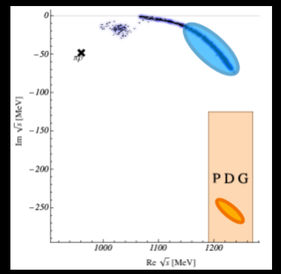
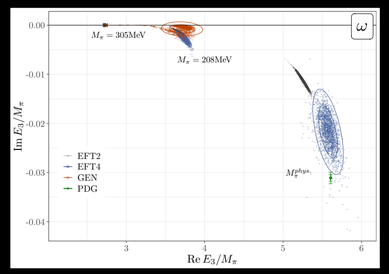
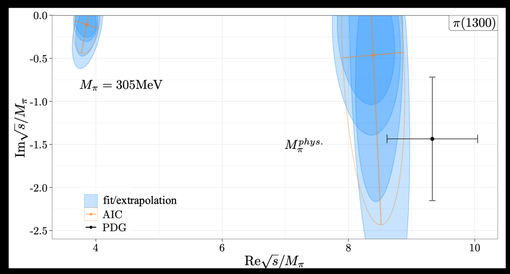
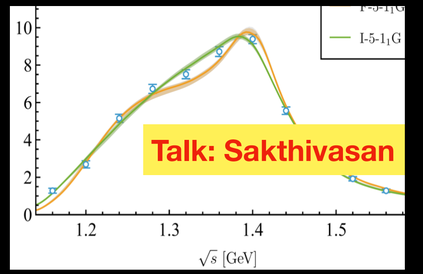








SUMMARY

OUTLOOK

it is just the beginning!

✳ Applications: $DD\pi$, $N(1440)$, . . . spin-exotics? — $a_1(1420)$

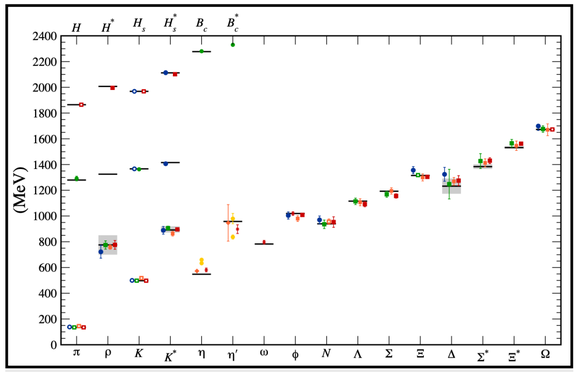
✳ FLAG ready: Systematics/statistics improvement, cutoff treatment…

✳ EFT tests: Universality of $\omega \to 3\pi, \rho \to 2\pi$ coupling, …

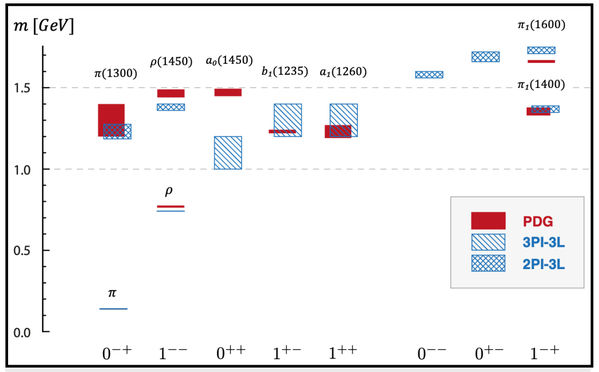
STATE OF THE ART

Theory

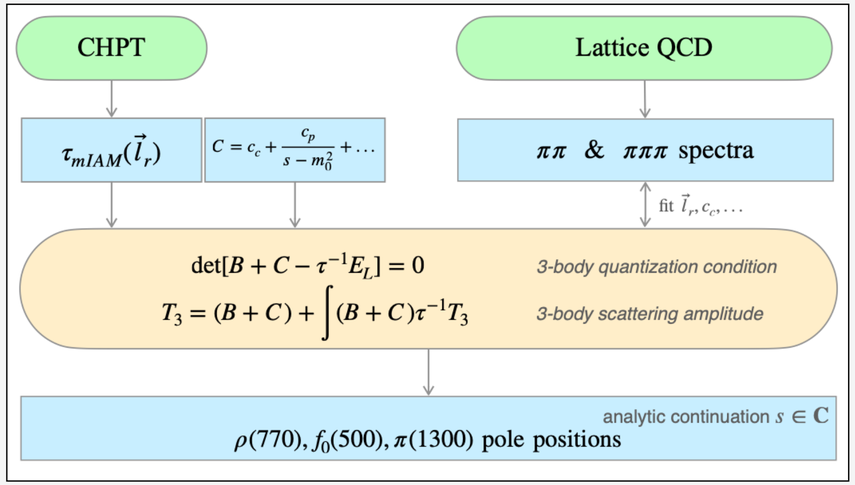
- Constituent quark model — Wang/Ping Chin. Phys. Lett. 24, 1195 (2007)
- DSE beyond rainbow-ladder cont. — Williams/Fischer Phys.Rev.D 93 (2016) 3, 034026
- Unitary models $\pi K \bar{K}$ — Phys.Rev.D 84 (2011) 074027
- Pioneering Lattice QCD (spectrum) calculations — Dudek+ (HadSpec) PRD 82, 034508 (2010); McNeile/Michael (UKQCD) PLB 642, 244 (2006)



Kronfeld Ann. Rev. Nucl.Part. Sci. 62, 265 (2012)



Eichmann+ Prog.Part.Nucl.Phys. 91 (2016)



This work – workflow

- Three-body operators/scattering states lattice QCD calc.
- Three-body finite-volume formalisms
- Pole positions
- Chiral extrapolations (CHPT) to the physical point

LATTICE SETUP

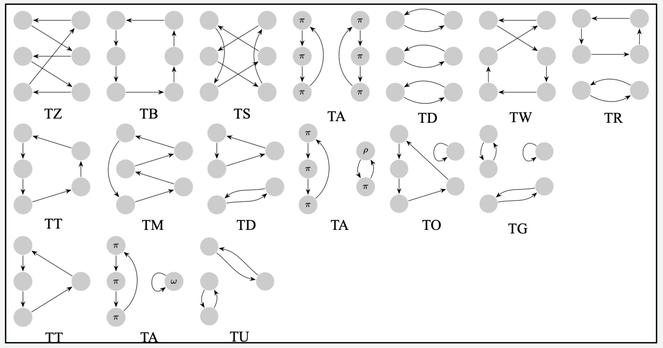
- 4 CLQCD ensembles with N=2+1 Clover fermions [CLQCD 2024]
  - Two volumes
  - Two pion masses (208 MeV, 305MeV)

configuration	volume	a/fm	m_π/MeV	N_{cfgs}
C48P14	$48^3 \times 96$	0.10530(18)	135.5(1.6)	259
F32P21	$32^3 \times 64$	0.07746(18)	210.9(2.2)	459
F48P21	$48^3 \times 96$	0.07746(18)	207.2(1.1)	222
F32P30	$32^3 \times 96$	0.07746(18)	303.2(1.3)	567
F48P30	$48^3 \times 96$	0.07746(18)	303.4(9)	201
H48P32	$48^3 \times 144$	0.05187(26)	317.2(0.9)	274

- **OpTion** utilized for 3-hadron operators with arbitrary momenta

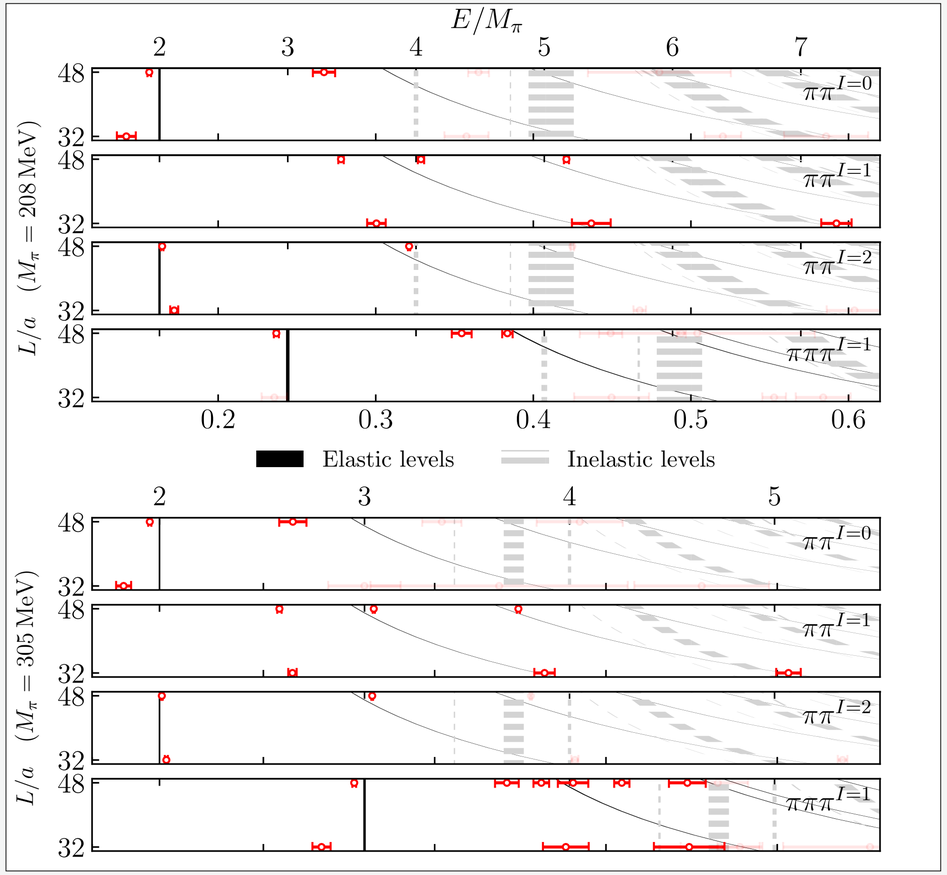
  Yan+ arXiv:2507.16070, JHEP (2025) (**https://github.com/wittscien/OpTion**)

- Contractions (~400)



Finite-volume spectrum:

24 ($\pi\pi$) + 12 ($\pi\pi\pi$) interacting levels



QUANTIZATION CONDITIONS

Lüscher QC

$$\sqrt{\sigma} \in \{E_0, E_1, \dots\}_{\pi\pi} \iff \tilde{K}^{-1}(\sigma) - \Sigma^L(\sigma) \equiv 0$$

Döring/Meißner/Rusetsky/Oset *Eur.Phys.J.A* 47 (2011) 139

- Equivalent to 2b-Lüscher QC $\mathcal{O}(e^{-ML})$
- Unknown 2-body interaction — use UCHPT

Hanhart+ PRL100, 152001 (2008)

$$\tilde{K}^{-1}(\sigma) \quad \to \quad T_{mIAM}[l_1, l_2, l_3, l_4](\sigma)$$

Finite Volume Unitarity (FVU) 3-body quantization condition*

$$\sqrt{s} \in \{E_0, E_1, \dots\}_{\pi\pi\pi} \iff \det\left[\, 2L^3 E_{\mathbf{p}} \left(\tilde{K}^{-1}(s) - \Sigma^L(s)\right) - B(s) - C(s) \,\right] \equiv 0$$

MM/Döring *Eur.Phys.J.A* 53 (2017) 12, 240

- Extended to all isospin/strangeness channels

Feng+ *Phys.Rev.D* 110 (2024) 094002
K(1460) 2511.02543

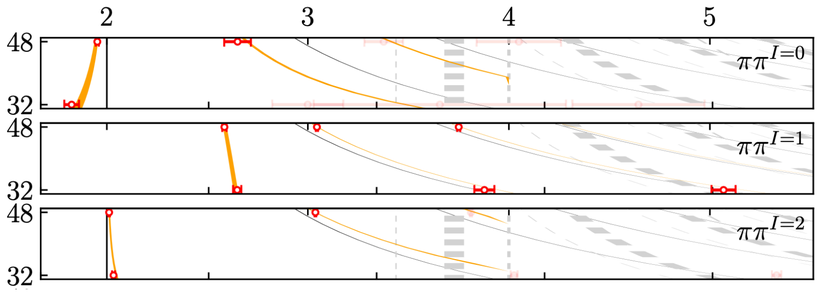
- Unknown 2-body interaction — use UCHPT

Hanhart+ PRL100, 152001 (2008)

$$\tilde{K}^{-1}(\sigma) \quad \to \quad T_{mIAM}[l_1, l_2, l_3, l_4](\sigma)$$

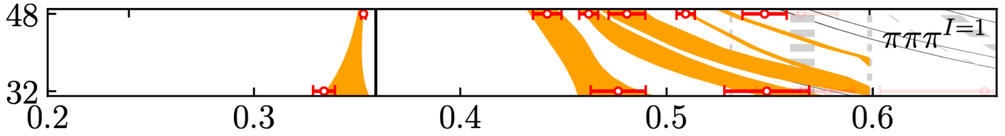
- Unknown 3-body force

$$C(s) \quad \to \quad c^{\alpha\beta} = c_c^{\alpha\beta} + \frac{c_p^{\alpha\beta}}{s - m_0^2}, \quad \alpha = \beta \in \{\sigma\pi, \rho\pi\}$$



2/3-body fit

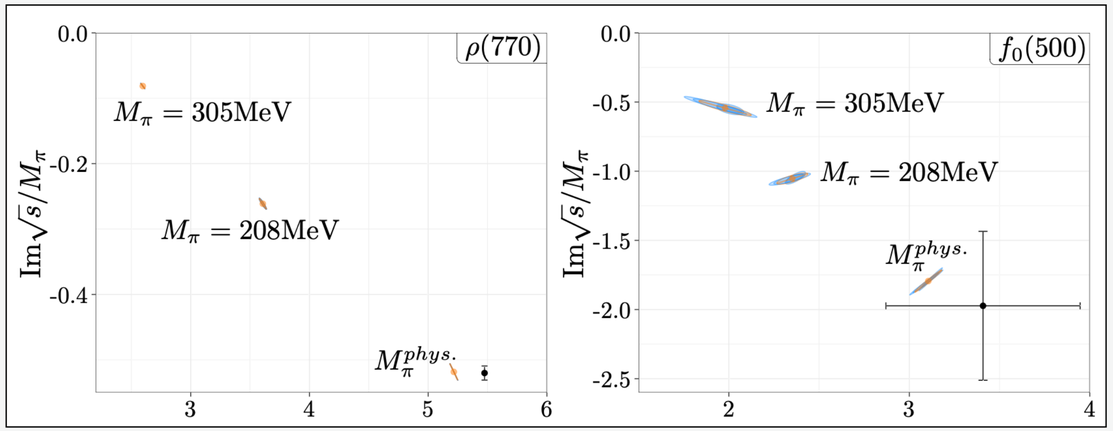
$$\langle c_c, c_p, m_0, l_1, l_2, l_3, l_4 \rangle$$



*) **Equivalent/not identical alternatives:** RFT, NREFT, …(Hansen/Sharpe 2014, Rusetsky/Hammer/Pang 2017, … Jackura et al. *Phys.Rev.D* 100 (2019) 3, 034508, Garofalo et al. *JHEP* 02 (2023) 252)
Developments and applications (Blanton, Romero-Lopez, Hansen, Briceño, Dawid, Rusetsky, Davoudi, Guo, MM, Culver, Yan, Garofalo, Urbach, Polejaeva, Raposo, Feng, Döring, …)
Reviews: Hansen/Sharpe Ann. Rev. Nucl. Part. Sci. 69, 65 (2019) MM/Doring/Rusetsky Eur. Phys. J. ST 230, 1623 (2021) Romero-Lopez, PoS LATTICE2022, 235 (2023))
Related talks: Raposo (Tcc/LHC), Sharpe (Npipi), Romero-Lopez (pipipi)

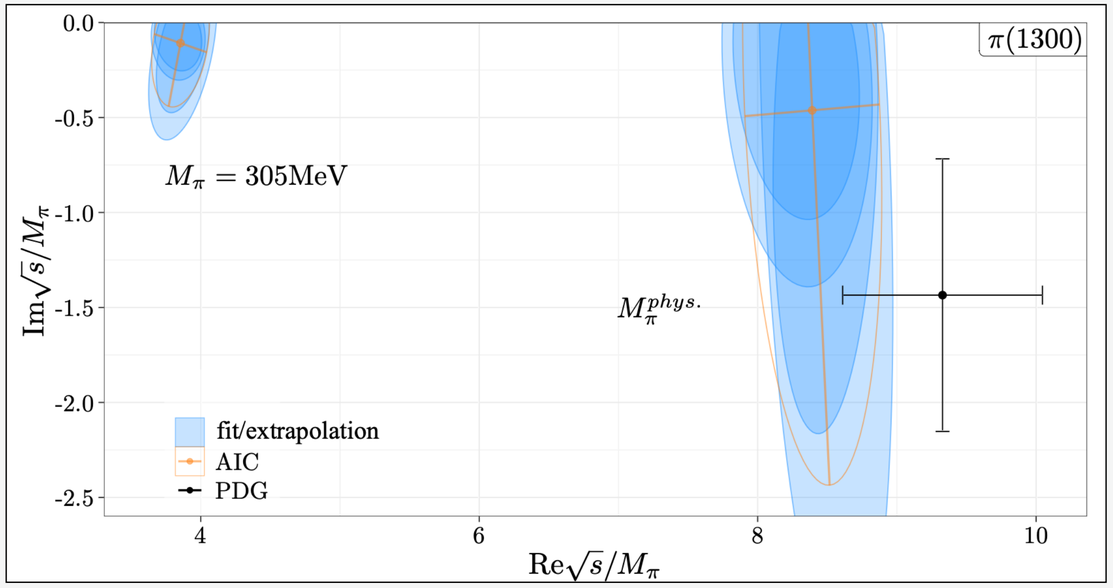
$\pi(1300)$ FROM LATTICE QCD

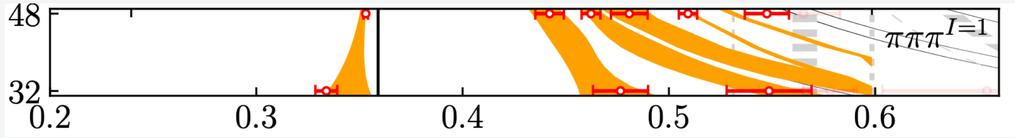
[agemo] Yan+ PRL136 (2026) no.14, 141901



2b resonances and chiral trajectories (in passing)

- $f_0(500)$ agrees with PDG
- $\text{Re}\, M_{\rho(770)}$ slightly too low
  - PDG value improved drastically Hoferichter+ PL B853 (2024)
  - Discretization effects? Over-constrained by mIAM?





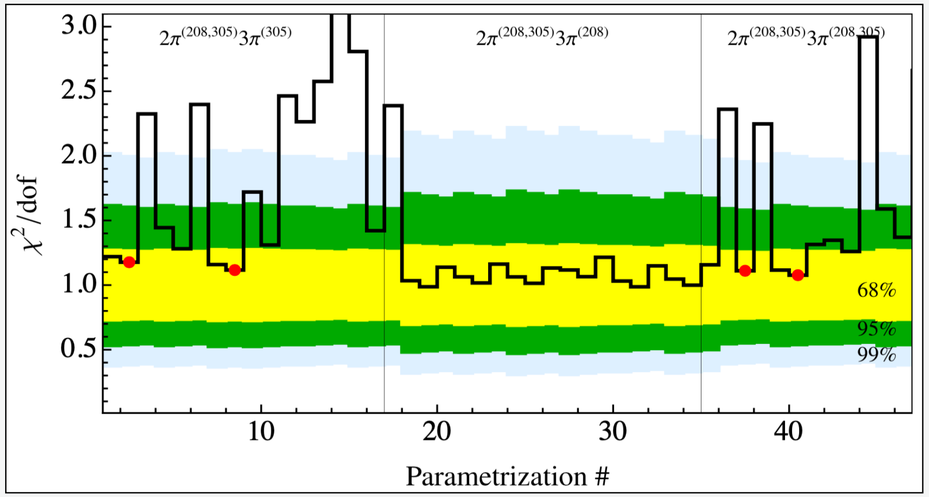
Excited state of the pion

- $M_\pi = 305\,\text{MeV}$ finite-volume ensemble
  - ~~no-resonance scenario~~
  - Includes *dynamically generated* scenarios
- Physical point extrapolation
  - Mass/Width agree with PDG (1σ)

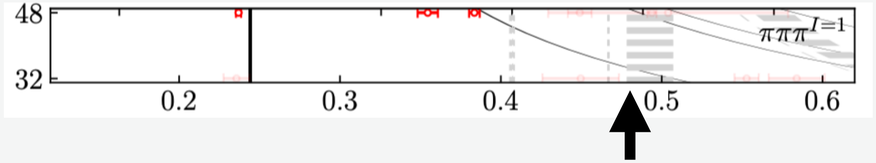
THINKING “OUTSIDE OF THE BOX”

Fits

- $\mathcal{O}(2000)$ fit variations $\langle c_c, c_p, m_0, l_1, l_2, l_3, l_4 \rangle$
- Best $\chi^2/\mathrm{dof} \approx 1.1$



- $\pi\pi\pi^{208}$ does not restrict $\pi(1300)$ region in the elastic regime



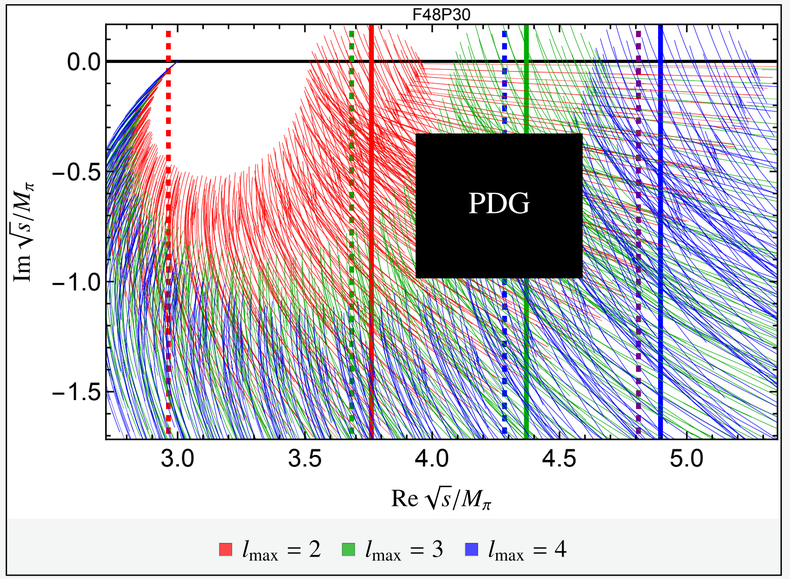
$M_{\pi(1300)}/M_\pi \approx 1300/208$

Infinite-volume scattering amplitudes

- $\tilde{K}^{-1}(s), C(s)$ are fixed and **volume independent**

$$T^c = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B+C)}{2E_l} \frac{1}{\tilde{K}^{-1} - \Sigma} T^c$$

- Maximize the analytically accessible area ($\ell_{\max}$, …)



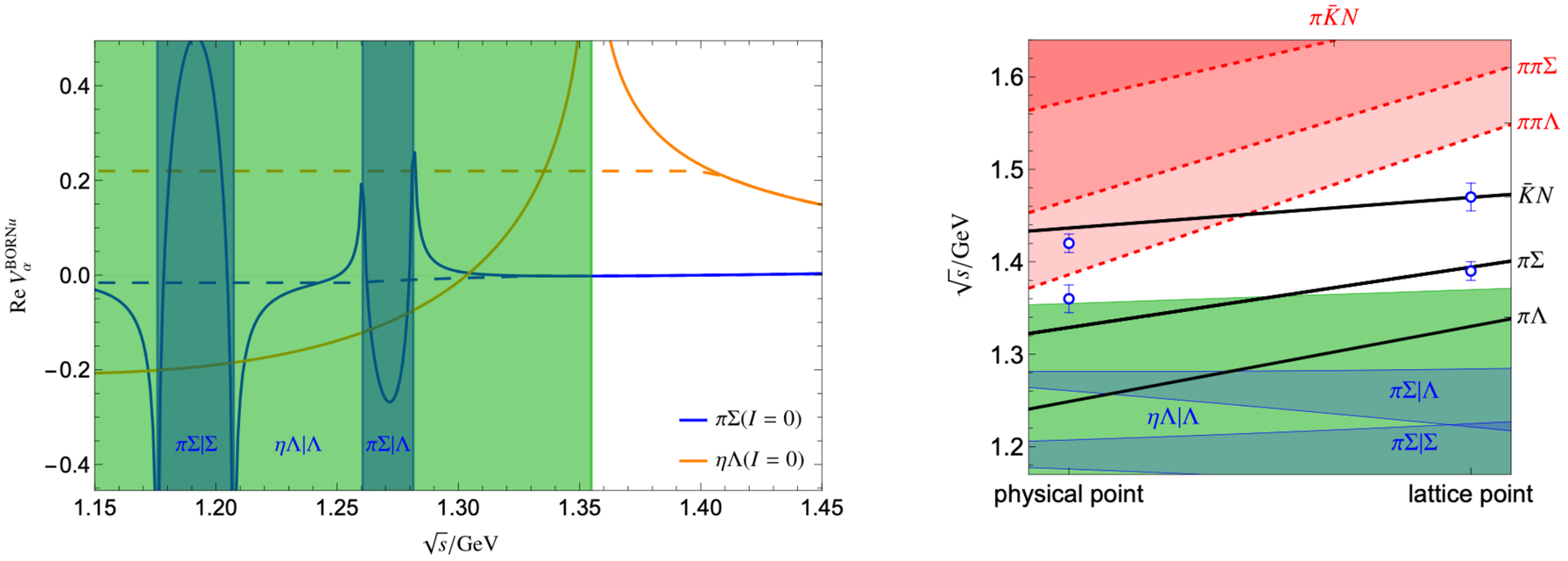
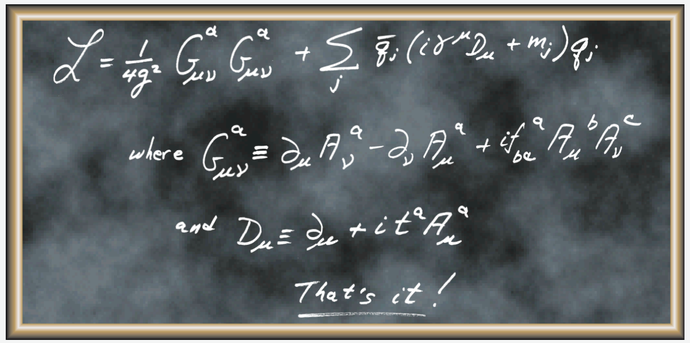


Figure 5. Three-body related singularities for the physical and unphysical quark-mass setups. Left: Singularities of the u-channel Born diagram Eq. (4.10) projected to $I = 0$ for the initial/final states as specified in the legend for physical quark masses. Long-dashed lines represent the potential implemented in the UCHPT amplitude mitigating the appearance of u-channel left-hand cuts. Right: Relevant short left-hand cut associated with the u-channel exchange (green, blue areas), c.f. left figure. Black solid and red dashed lines denote the position of the right-hand cut branching points with respect to two- and three-body states, respectively. Energy region with no allowed three-body onshell states is the remaining white area. Blue dots with error bars denote the averaged result from the global analysis discussed in Sect. VI.

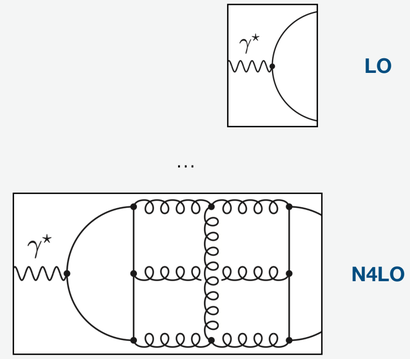
THEORY



www.frankwilczek.com/Wilczek_Easy_Pieces/298_QCD_Made_Simple.pdf

Quantum Chromodynamics (QCD)

- Degrees of freedom: quarks and gluons
- pQCD: well-defined set of rules (Feynman diagrams) to calculate transition rates

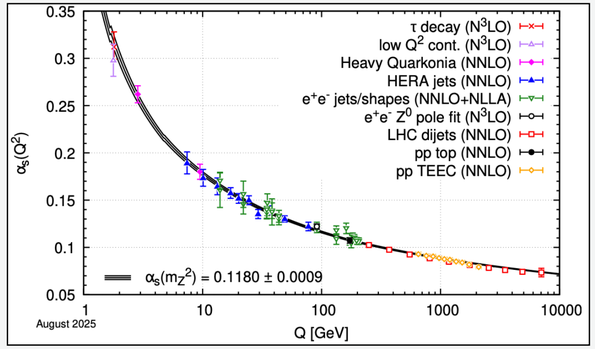


JHEP 08 (2017) 113

Hadron spectroscopy

- Virtual effects: α_s depends on the momenta
  - $\alpha_s(Q) > 1$ *for small exchanged momenta*
  - *asymptotic states = stable hadrons* $p, n, \pi, \ldots$

←Spectroscopy pQCD→



Particle Data Group 2025

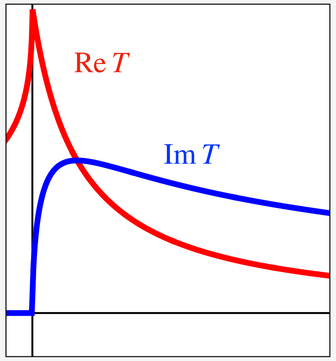
QUANTIZATION CONDITION: 2-BODY EXAMPLE

Review: MM/Doring/Rusetsky Eur.Phys.J.ST 230 (2021)

Continuum QFT

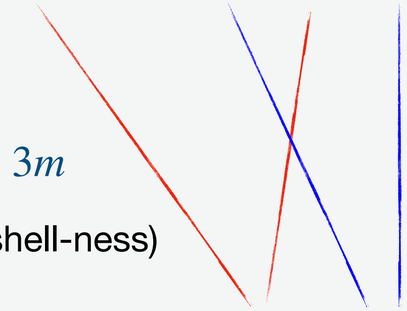
- Asymptotic states at $t \to \pm\infty$
- underlying quantity: $(\sigma \sim |S|^2, \dots)$

$$S = 1 + iT \in \mathbb{C}$$



$$\text{disc}\, T \sim |T|^2 \quad \sqrt{s} < 3m$$

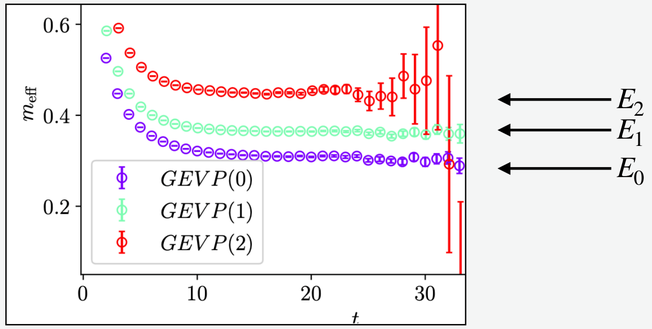
unitarity condition (on-shell-ness)



$$T^{-1} = K^{-1} - \int_l \frac{1}{2E_l}\frac{1}{(s - 4E_l^2 + i\epsilon)} = p\cot\delta - \left(\int \dots - Re\int \dots\right)$$

Finite-volume setup

- Input: energy eigenvalues $\{E_0, \dots\} \in \mathbb{R}$
- Correlation functions at $it \to \infty$



- off-shell configurations decay exponentially $\sim O(e^{-ML})$
- on-shell states propagate/feel box-size $\sim O((ML)^n)$

$$T_{FV}^{-1} = p\cot\delta - \left(\frac{1}{L^3}\sum_{\vec{l}\in 2\pi/L\mathbb{Z}} \dots - Re\int_{\vec{l}} \dots\right) = 0$$

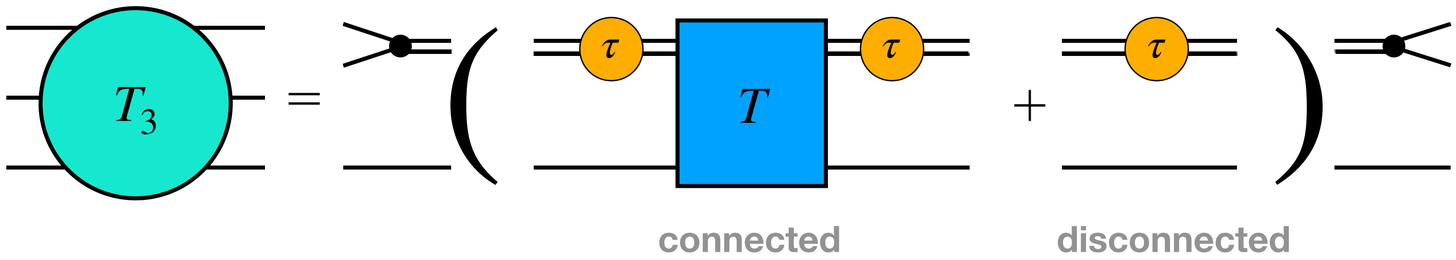
$$p\cot\delta(E_i) = Z_{00}(E_i) + \mathcal{O}(e^{-ML})$$

M. Lüscher, Nucl. Phys. B 354, 531 (1991)

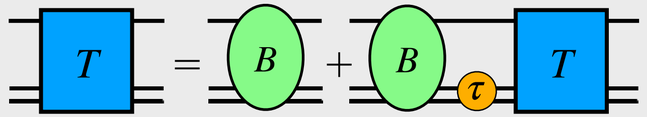
3-BODY INTERACTIONS

… same logic: start with a general scattering amplitude

MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)



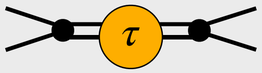
Bethe-Salpeter ansatz



… 4D integral equation

… unknown building blocks

isobar, aka **dimer**, aka **2-body subsystem**



- a tower of states for $L = 0,1,2,..$
- can be repulsive (!)

MM/Doring Phys.Rev.Lett. 122 (2019)

… imposing 3-body unitarity

Olive/Eden/… “The Analytic S-matrix”
Bedaque, Griesshammer (1999)
R. Aaron, R. D. Amado, and J. E. Young, Phys.Rev. 174, 2022 (1968)

3-BODY UNITARITY

… imposing 3-body unitarity

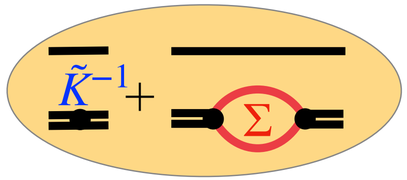
$$\langle\{q\}\,|\,(\hat{T}_3-\hat{T}_3^\dagger)\,|\,\{p\}\rangle = i\int\prod_{\ell=1}^{3}\frac{\mathrm{d}^4k_\ell}{(2\pi)^4}\,(2\pi)\delta^+(k_\ell^2-m^2)\,(2\pi)^4\delta^4\left(P-\sum_{\ell=1}^{3}k_\ell\right)\langle\{q\}\,|\,\hat{T}_3^\dagger\,|\,\{k\}\rangle\,\langle\{k\}\,|\,\hat{T}_3\,|\,\{p\}\rangle$$

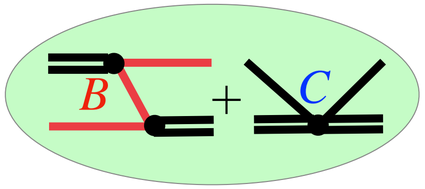
… exact solution — constraints on $\mathrm{disc}\,B$ and $\mathrm{disc}\,\tau^{-1}$

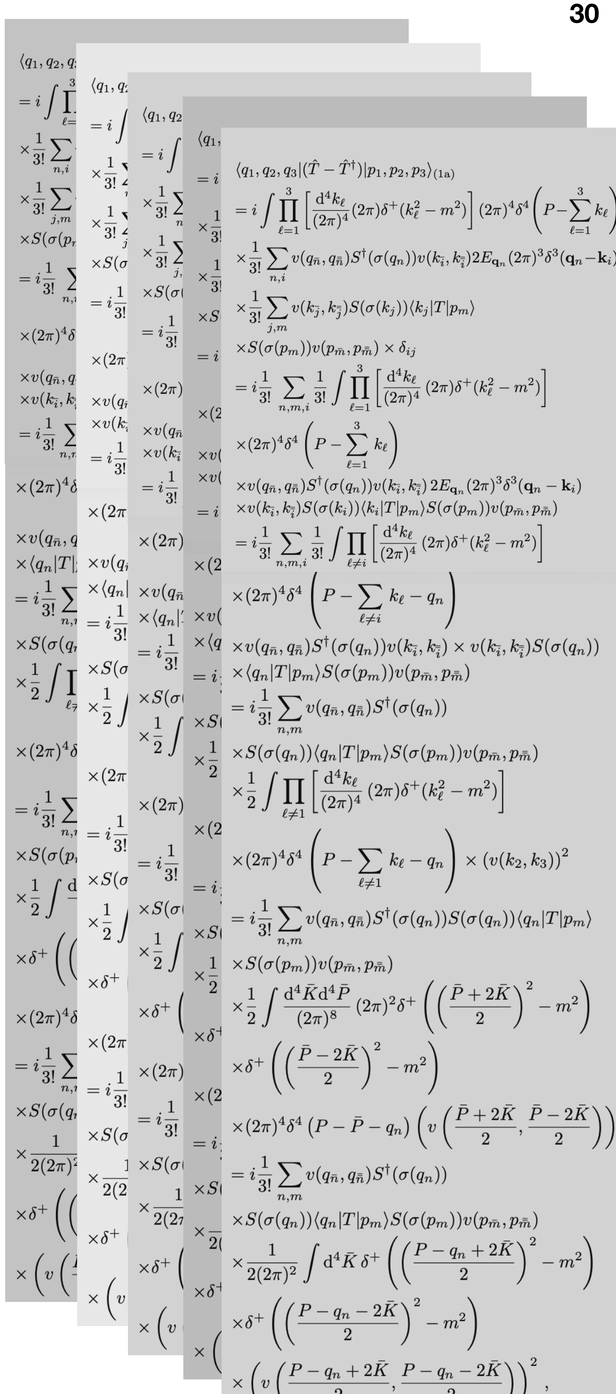
MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)

$$\tau^{-1}(\sigma(k)) = A + B\,\sigma(k) + \frac{\sigma(k)^2}{\pi}\int_{4m^2}^{\infty} d\sigma'\,\frac{\mathrm{Im}\,\tau^{-1}(\sigma')}{\sigma'^2(\sigma'-\sigma(k)-i\epsilon)}$$

$$B(s,p,p') = -\frac{v(Q,q)v(Q,p)}{2\sqrt{m^2+\mathbf{Q}^2}\left(E_Q-\sqrt{m^2+\mathbf{Q}^2}+i\epsilon\right)}$$



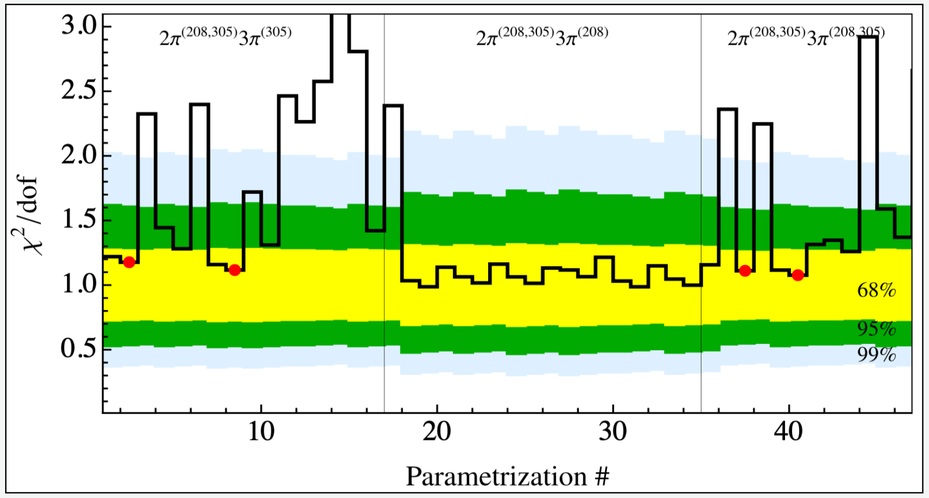




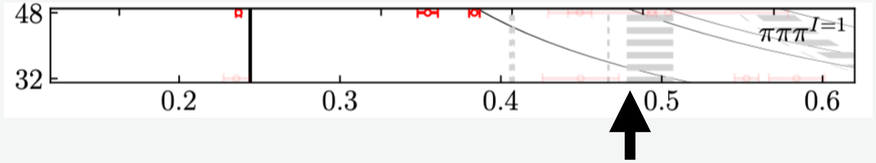
THINKING "OUTSIDE OF THE BOX"

Fits

- $\mathcal{O}(2000)$ fit variations $\langle c_c, c_p, m_0, l_1, l_2, l_3, l_4 \rangle$
- Best $\chi^2/\mathrm{dof} \approx 1.1$



- $\pi\pi\pi^{208}$ does not restrict $\pi(1300)$ region in the elastic regime

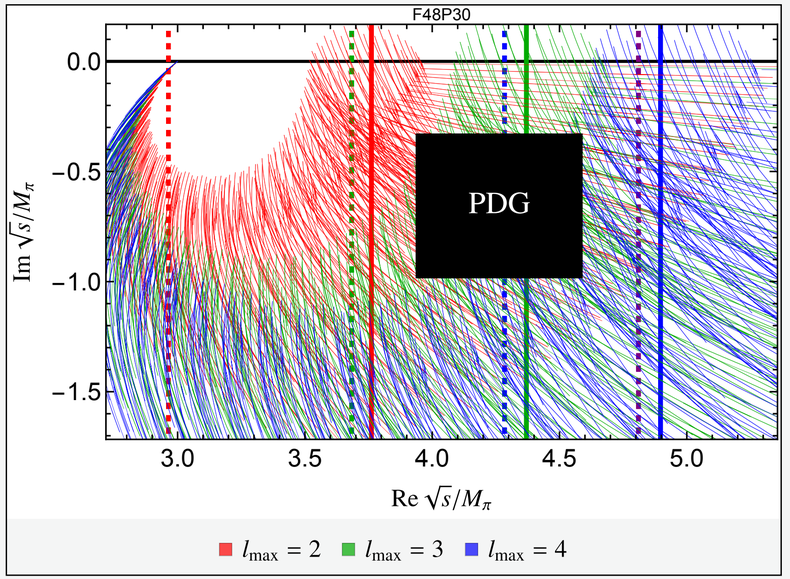


Infinite-volume scattering amplitudes

- $\tilde{K}^{-1}(s), C(s)$ are fixed and **volume independent**

$$T^c = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B+C)}{2E_l} \frac{1}{\tilde{K}^{-1} - \Sigma} T^c$$

- Maximize the analytically accessible area ($\ell_{\max}$, …)

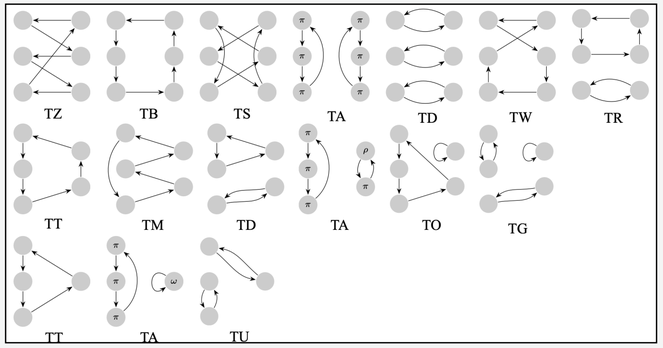


LATTICE SETUP

- 4 CLQCD ensembles with N=2+1 Clover fermions [CLQCD 2024]
  - Two volumes
  - Two pion masses (208 MeV, 305MeV)

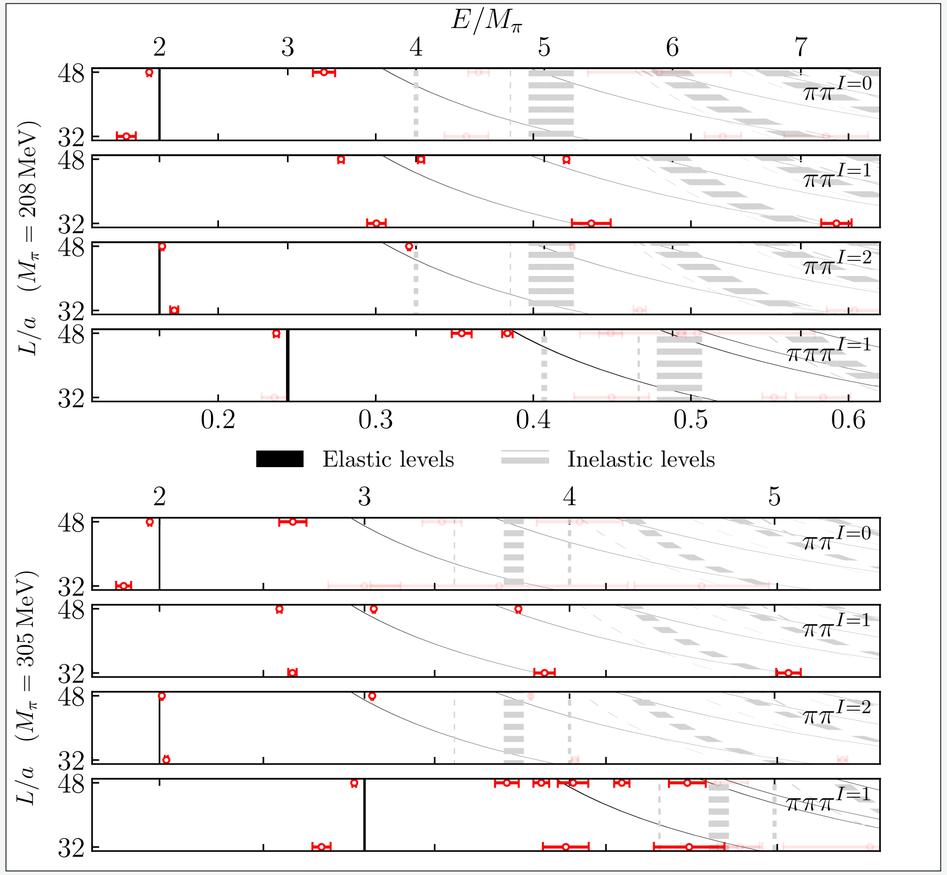
configuration	volume	a/fm	m_π/MeV	N_{cfgs}
C48P14	$48^3 \times 96$	0.10530(18)	135.5(1.6)	259
F32P21	$32^3 \times 64$	0.07746(18)	210.9(2.2)	459
F48P21	$48^3 \times 96$	0.07746(18)	207.2(1.1)	222
F32P30	$32^3 \times 96$	0.07746(18)	303.2(1.3)	567
F48P30	$48^3 \times 96$	0.07746(18)	303.4(9)	201
H48P32	$48^3 \times 144$	0.05187(26)	317.2(0.9)	274

- **OpTion** utilized for 3-hadron operators with arbitrary momenta
  Yan+ arXiv:2507.16070, JHEP (2025) (**https://github.com/wittscien/OpTion**)
- Contractions (~400)



Finite-volume spectrum:

24 ($\pi\pi$) + 12 ($\pi\pi\pi$) interacting levels



QUANTIZATION CONDITIONS

Lüscher QC

$$\sqrt{\sigma} \in \{E_0, E_1, \dots\}_{\pi\pi} \iff \tilde{K}^{-1}(\sigma) - \Sigma^L(\sigma) \equiv 0$$

Döring/Meißner/Rusetsky/Oset *Eur.Phys.J.A* 47 (2011) 139

- Equivalent to 2b-Lüscher QC $\mathcal{O}(e^{-ML})$
- Unknown 2-body interaction — use UCHPT

Hanhart+ PRL100, 152001 (2008)

$$\tilde{K}^{-1}(\sigma) \quad \to \quad T_{mIAM}[l_1, l_2, l_3, l_4](\sigma)$$

Finite Volume Unitarity (FVU) 3-body quantization condition*

$$\sqrt{s} \in \{E_0, E_1, \dots\}_{\pi\pi\pi} \iff \det\left[\, 2L^3 E_{\mathbf{p}} \left(\tilde{K}^{-1}(s) - \Sigma^L(s)\right) - B(s) - C(s) \,\right] \equiv 0$$

MM/Döring *Eur.Phys.J.A* 53 (2017) 12, 240

- Extended to all isospin/strangeness channels

Feng+ *Phys.Rev.D* 110 (2024) 094002
K(1460) 2511.02543

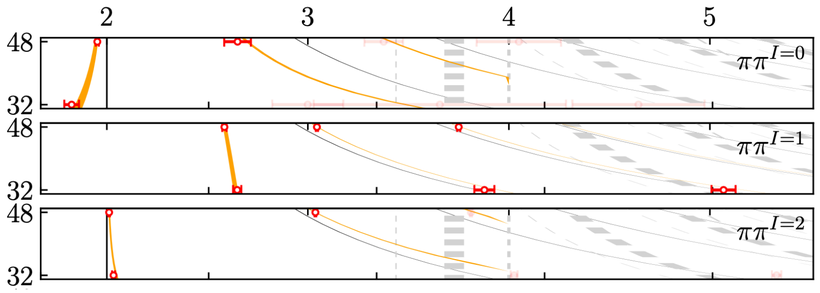
- Unknown 2-body interaction — use UCHPT

Hanhart+ PRL100, 152001 (2008)

$$\tilde{K}^{-1}(\sigma) \quad \to \quad T_{mIAM}[l_1, l_2, l_3, l_4](\sigma)$$

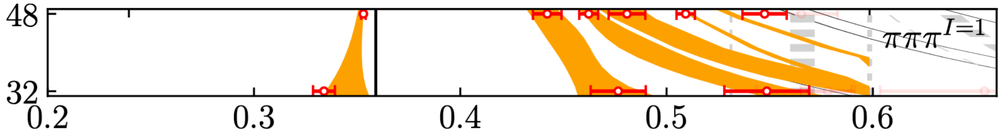
- Unknown 3-body force

$$C(s) \quad \to \quad c^{\alpha\beta} = c_c^{\alpha\beta} + \frac{c_p^{\alpha\beta}}{s - m_0^2}, \quad \alpha = \beta \in \{\sigma\pi, \rho\pi\}$$



2/3-body fit

$$\langle c_c, c_p, m_0, l_1, l_2, l_3, l_4 \rangle$$

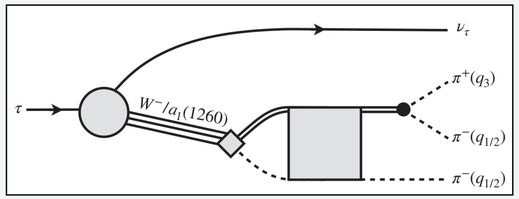


*) **Equivalent/not identical alternatives:** RFT, NREFT, …(Hansen/Sharpe 2014, Rusetsky/Hammer/Pang 2017, … Jackura et al. *Phys.Rev.D* 100 (2019) 3, 034508, Garofalo et al. *JHEP* 02 (2023) 252)
Developments and applications (Blanton, Romero-Lopez, Hansen, Briceño, Dawid, Rusetsky, Davoudi, Guo, MM, Culver, Yan, Garofalo, Urbach, Polejaeva, Raposo, Feng, Döring, …)
Reviews: Hansen/Sharpe Ann. Rev. Nucl. Part. Sci. 69, 65 (2019) MM/Doring/Rusetsky Eur. Phys. J. ST 230, 1623 (2021) Romero-Lopez, PoS LATTICE2022, 235 (2023))
Related talks: Raposo (Tcc/LHC), Sharpe (Npipi), Romero-Lopez (pipipi)

AXIAL-VECTOR MESON

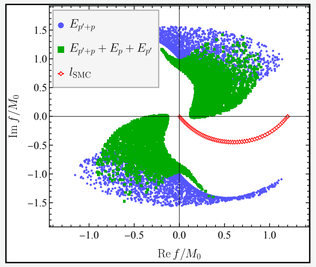
Sadasivan/MM/Culver/Alexandru/Lee/Doring Phys.Rev.D 101 (2020);

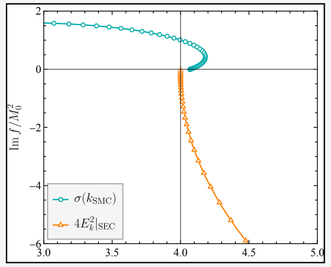
Input from τ-decays or hadronic reactions



Solution techniques

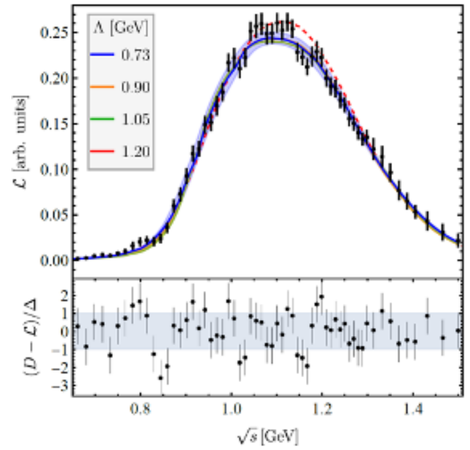
- avoid singularities (OPE, self-energy)
- complex contour deformation $|\ell| \in \mathbb{C}$

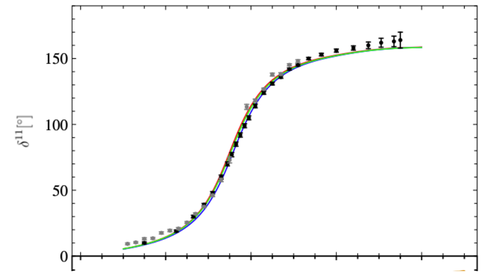




- Analytic continuation to the real axis

J. H. Hetherington and L. H. Schick, Phys. Rev. 137, B935–B948 (1965).
R. T. Cahill and I. H. Sloan, Nucl. Phys. A 165, 161–179 (1971)
W. Schmid and Ziegelmann,
S. K. Adhikari and R. D. Amado, Phys. Rev. D 9, 1467–1475 (1974)

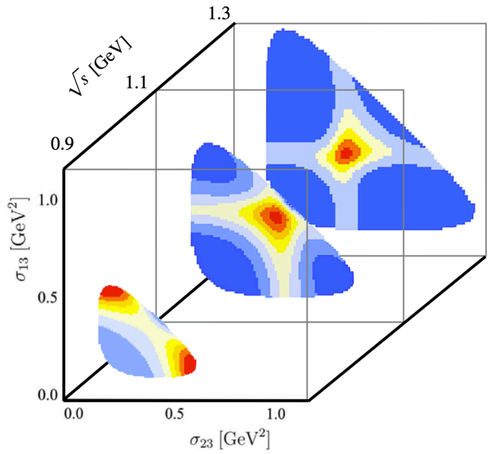


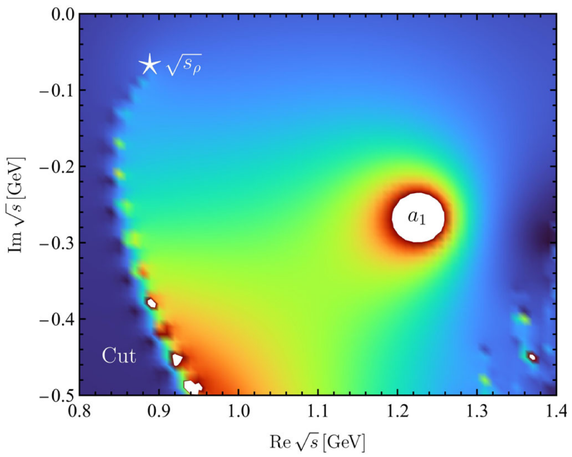


Schael et al. [ALEPH] Phys.Rept. 421 (2005);
Nucl.Phys.B 79; Phys.Rev.D 7;

IVU

$$T(s,p,p') = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B+C)}{2E_l} \frac{1}{\tilde{K}^{-1} - \Sigma} T$$





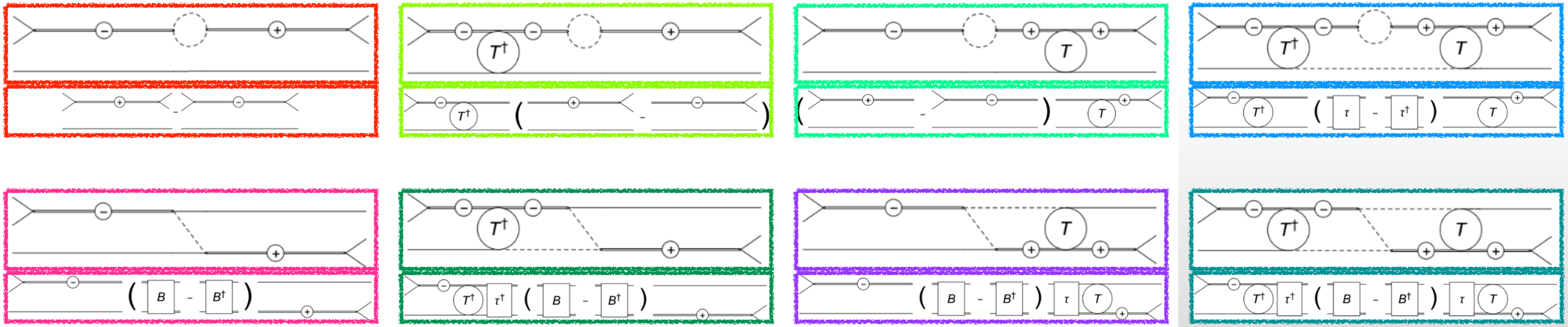
THREE-BODY UNITARITY

$$\mathcal{S} := \mathbb{1} + i(2\pi)^4\delta^4(P_i - P_f)\,\hat{T}$$

$$\langle q_1,q_2,q_3|(\hat{T}-\hat{T}^\dagger)|p_1,p_2,p_3\rangle = i\int\prod_{\ell=1}^{3}\left[\frac{\mathrm{d}^4k_\ell}{(2\pi)^4}(2\pi)\delta^+(k_\ell^2-m^2)\right](2\pi)^4\delta^4\left(P-\sum_{\ell=1}^{3}k_\ell\right)\langle q_1,q_2,q_3|\hat{T}^\dagger|k_1,k_2,k_3\rangle\,\langle k_1,k_2,k_3|\hat{T}|p_1,p_2,p_3\rangle$$

Exact identification

MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)



these are the ones of
R. Aaron, R. D. Amado, and J. E. Young
Phys.Rev. 174, 2022 (1968)

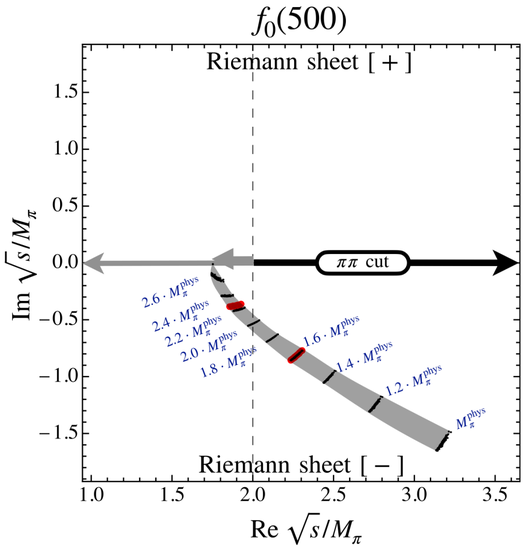
APPLICATION I

Two pion system

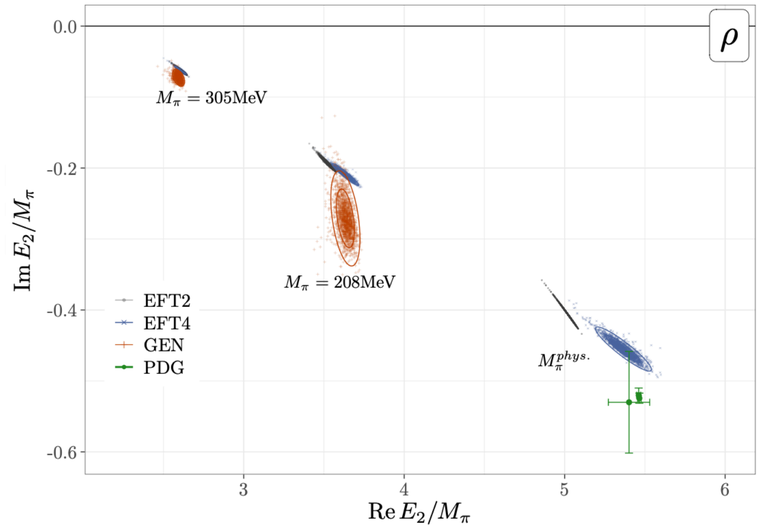
- simplest 2-hadron system
- many LQCD results

  NPLQCD; HadSpec; ETMC; GW-lattice; CP-PACS;….
- simultaneous description of all $\pi\pi$ interaction channels through CHPT — UCHPT

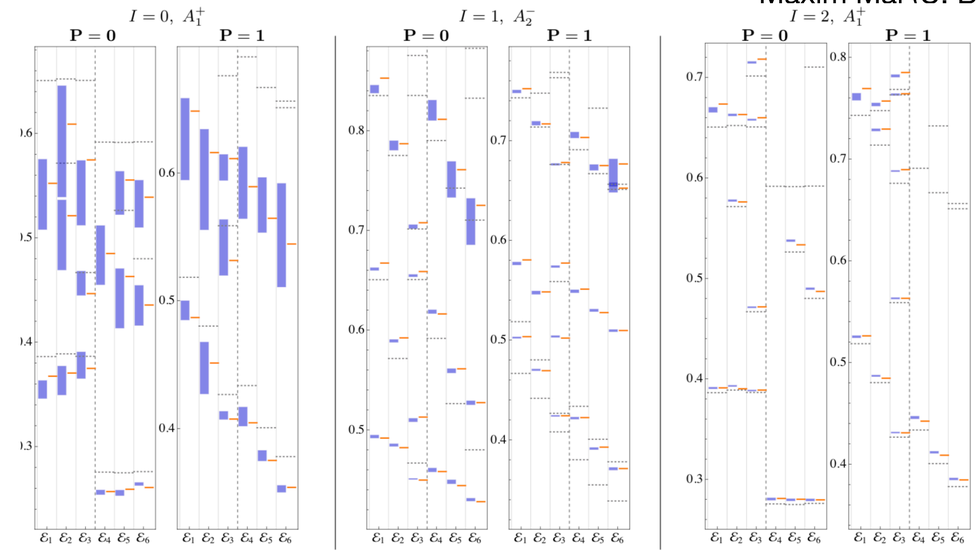
  GWQCD: Guo et al. (2016) Guo et al. (2018) Culver et al. (2019) MM et al.(2019)

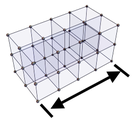


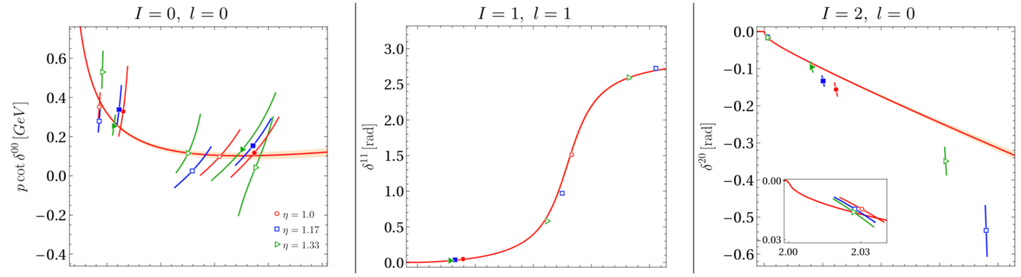
MM/Urbach/Meißner *Phys.Rept.* 1001 (2023) 1-66



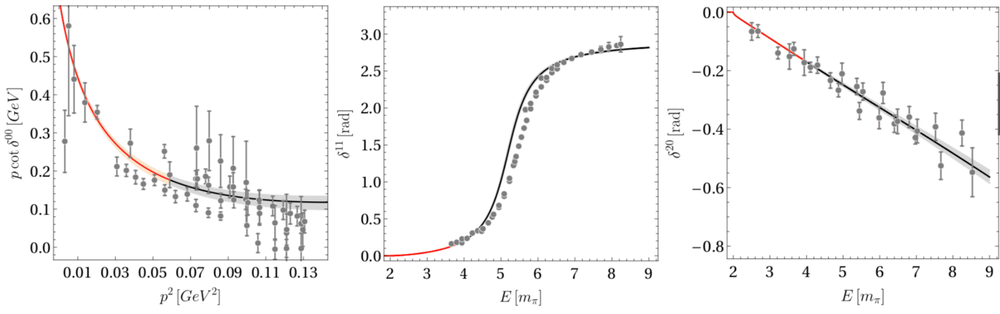
Yan/MM/… Phys.Rev.Lett. 133 (2024) 21







Quantization Condition



Chiral extrapolation

[GWQCD] MM/Culver/… Phys.Rev.D100(11)(2019) 114514

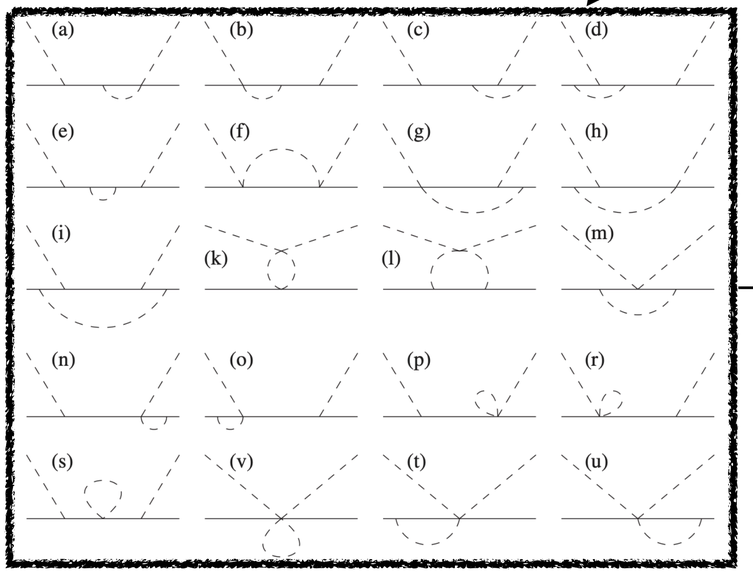
APPLICATIONS I

Meson-baryon scattering from CHPT

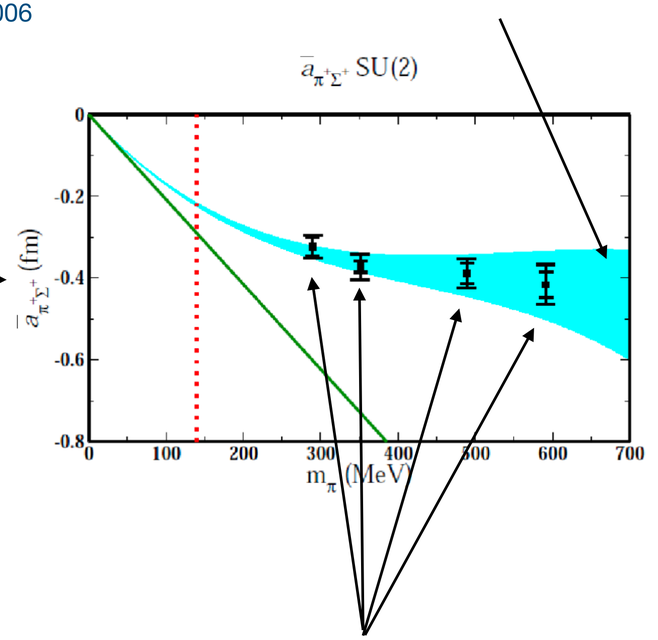
MM/P.C.Bruns/Ulf-G. Meißner/B.Kubis Phys.Rev.D 80 (2009) 094006

- full SU(3) dynamics near threshold
- agrees with experiment in many cases
- well controlled chiral extrapolation of Lattice QCD results ($\pi\Sigma$)

$$\mathcal{L}^{(2)}_{\phi B} = b_{D/F}\langle\bar{B}[\chi_+, B]_\pm\rangle + b_0\langle\bar{B}B\rangle\langle\chi_+\rangle + b_{1/2}\langle\bar{B}[u_\mu, [u^\mu, B]_\mp]\rangle + b_3\langle\bar{B}\{u_\mu, \{u^\mu, B\}\}\rangle + b_4\langle\bar{B}B\rangle\langle u_\mu u^\mu\rangle$$
$$+ i\sigma^{\mu\nu}(b_{5/6}\langle\bar{B}[[u_\mu, u_\nu], B]_\mp\rangle + b_7\langle\bar{B}u_\mu\rangle\langle u_\nu B\rangle) + \frac{ib_{8/9}}{2m_0}(\langle\bar{B}\gamma^\mu[u_\mu, [u_\nu, [D^\nu, B]]_\mp]\rangle + \langle\bar{B}\gamma^\mu[D_\nu, [u^\nu, [u_\mu, B]]_\mp]\rangle)$$
$$+ \frac{ib_{10}}{2m_0}(\langle\bar{B}\gamma^\mu\{u_\mu, \{u_\nu, [D^\nu, B]\}\}\rangle + \langle\bar{B}\gamma^\mu[D_\nu, \{u^\nu, \{u_\mu, B\}\}]\rangle) + \frac{ib_{11}}{2m_0}(2\langle\bar{B}\gamma^\mu[D_\nu, B]\rangle\langle u_\mu u^\nu\rangle$$
$$+ \langle\bar{B}\gamma^\mu B\rangle\langle[D_\nu, u_\mu]u^\nu + u_\mu[D_\nu, u^\nu]\rangle), \qquad (8)$$



MM/P.C.Bruns/Ulf-G. Meißner/B.Kubis Phys.Rev.D 80 (2009) 094006



LQCD at unphysical pion masses
Torok/Beane/Detmold/Luu/… Phys.Rev.D 81 (2010) 074506

AXIAL-VECTOR MESON*

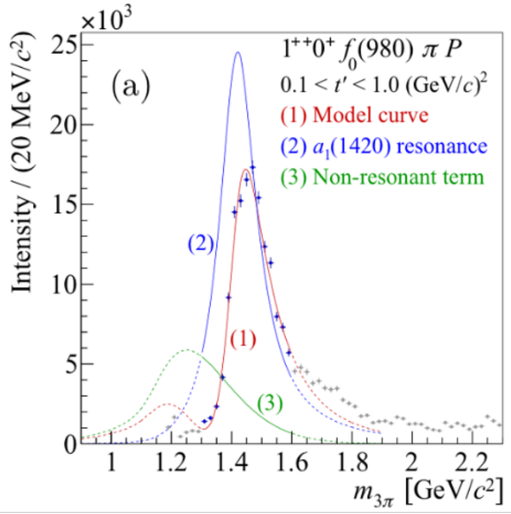
Excited axial-vector meson: $a_1(1420)$

Observed by COMPASS/Belle in $\pi^-\pi^+\pi^-$ final state

Creation mechanisms: COMPASS:2015kdx, Rabusov:2023tna

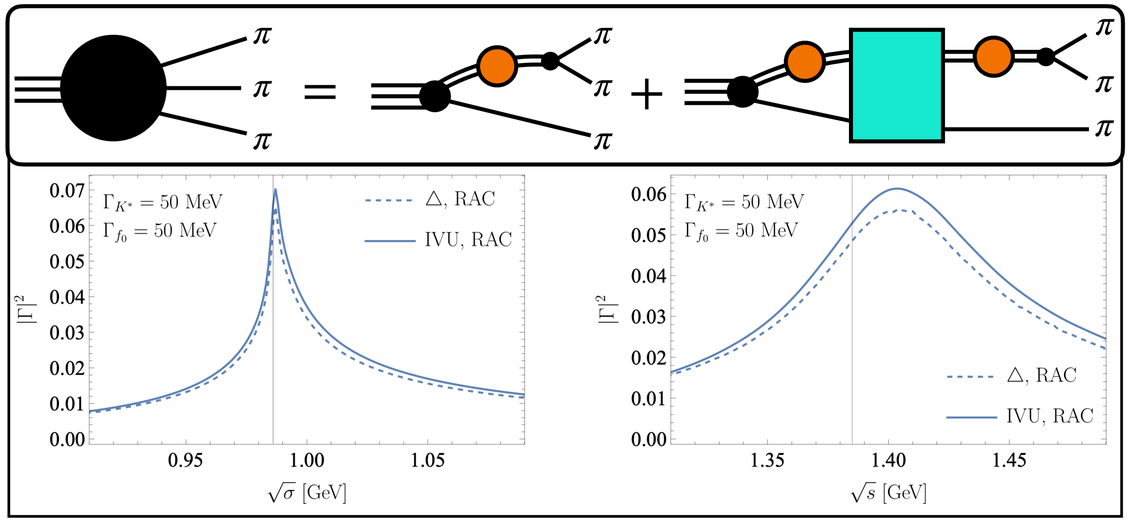
- Excited state of $a_1(1260)$
- "Triangle singularity" $K^*(892)\bar{K} \xrightarrow{K} \pi f_0(980)$ COMPASS:2020yhb

Mikhasenko:2015oxp **Review: Guo:2019twa**
Related: Dai:2018hqb, Dai:2018rra, Liang:2019jtr, Jing:2019cbw, Du:2021zdg, Duan:2023dky, Wang:2016dtb, Nakamura:2023obk, Zhang:2024dth, Achasov:2022onn, Nakamura:2023hbt, arXiv:1609.04133 [hep-ph].
Talks: J.J.Wu — Z.Zhang



3b unitary formalism IVU Ajay Sakthivasan/MM/Döring/Rusetsky 2407.17969 [hep-ph]

- formalism to incorporate both hypothesis
- full 2- and 3-body re-scattering
- **for now:** only kinematic/analytical properties (no spin)



➡ Effect is small but distinguishable

➡ Add spin, fit to the line-shapes … in progress

VECTOR MESON

Mapping to infinite volume

- 3-body quantization condition

FVU

$$\det\left[\, 2L^3 E_{\mathbf{p}} \left(\tilde{K}^{-1} - \Sigma^L\right) - B - C \,\right]^{\Lambda} \equiv 0$$

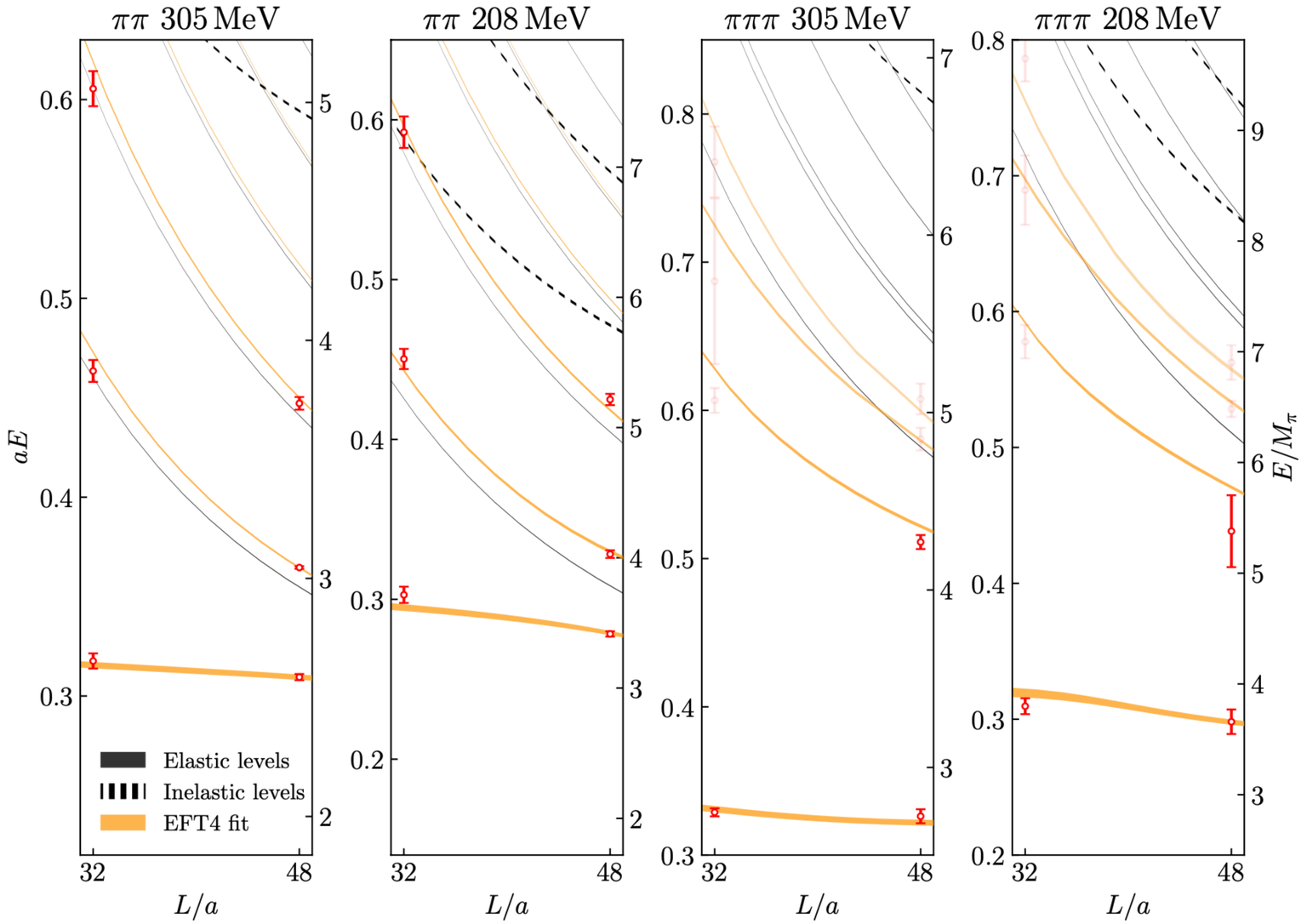
MM/Döring
Eur.Phys.J.A 53 (2017) 12,

- Volume-independent 2-,3-body force $C, \tilde{K}$
- saturated by meson s-channel interaction — **EFT form**

$$\left[\tilde{K}^{-1}\right]_{\boldsymbol{p}'\lambda',\boldsymbol{p}\lambda} = \delta_{\lambda'\lambda}\delta_{\boldsymbol{p}'\boldsymbol{p}}\frac{\sigma_p - M_\rho^2}{2g^2}\,,$$

$$\tilde{c}_{11} = \frac{6s(M_\rho^2 - \sigma_q + 6g^2 f_\pi^2)(M_\rho^2 - \sigma_p + 6g^2 f_\pi^2)}{64g^2\pi^3 f_\pi^6 (s - M_\omega^2)}\,,$$

Finite-volume spectrum = Energy eigenvalues



H.Yan/MM/Garofalo/Meißner/Lui/Liu/Urbach: 2407.16659 [hep-lat]

Current frontier: 3-body dynamics from LQCD

➥ 3-body Quantization Conditions1

➥ RFT / FVU / NREFT

➥ many perturbatively interacting systems are studied2

$$0 = \det\left(L^3 \left(\tilde{F}/3 - \tilde{F}(\tilde{K}_2^{-1} + \tilde{F} + \tilde{G})^{-1}\tilde{F} \right)^{-1} + K_{\mathrm{df},3} \right)$$ RFT

$$0 = \det\left(B_0 + C_0 - E_L \left(K^{-1}/(32\pi) + \Sigma_L \right) \right)$$ FVU

3-body force

2-body interaction

one-particle exchange

2-body self-energy

1) Rusetsky, Bedaque, Grießhammer, Sharpe, Meißner, Döring, Hansen, Davoudi, Guo....
Reviews:
Hansen/Sharpe Ann.Rev.Nucl.Part.Sci. 69 (2019);
MM/Döring/Rusetsky Eur.Phys.J.ST 230 (2021);
2) MM/Döring PRL122(2019); Blanton et al. PRL 124 (2020); Hansen et al. PRL 126 (2021); ....

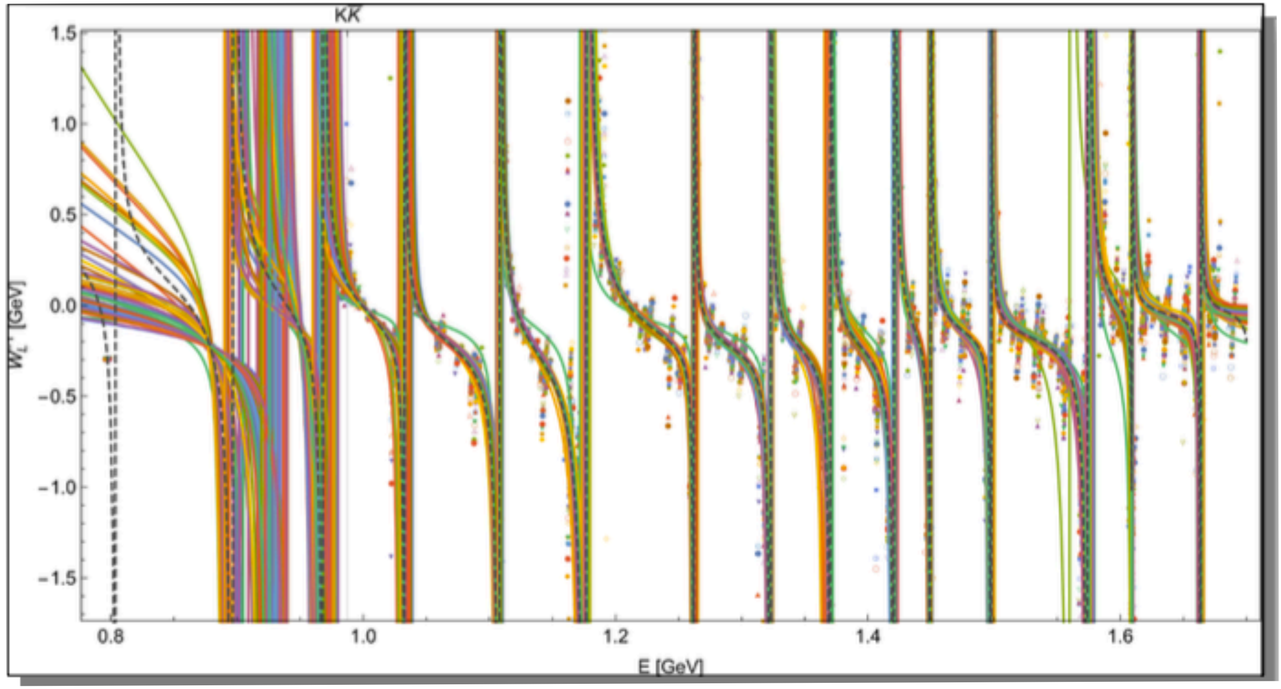
OPTICAL POTENTIAL

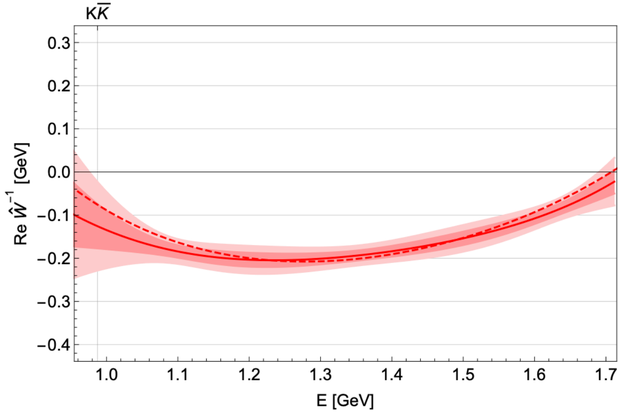
Alternatively: bulk properties of int. spectra

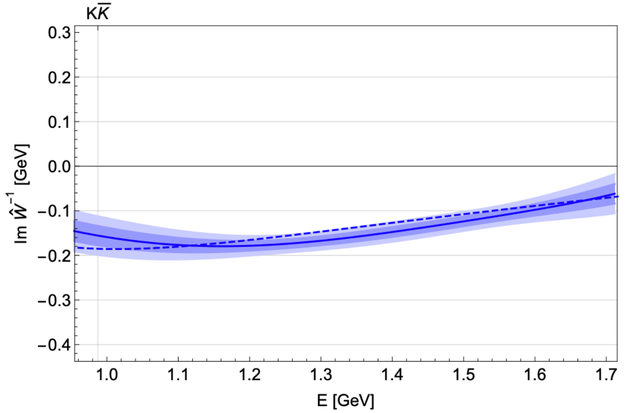
- double-limit prescription1

$$W^{-1}(E) = \lim_{\epsilon \to 0} \lim_{L \to \infty} W_L^{-1}(E + i\epsilon)$$

- smoothing and inverse problem2
- typically many input (EEVs) required







Agadjanov/MM/.. JHEP 06 (2016)

1) DeWitt, Phys. Rev. 103 (1956)
2) Agadjanov/MM/.. JHEP 06 (2016); Hansen et al. Phys.Rev.D 96 (2017) Bulava/Hansen Phys.Rev.D 100 (2019); Briceño et al. Phys.Rev.D 103 (2021)

EXAMPLE $\Lambda(1405) \ldots \Lambda(1380)$

Long history of experimental and theoretical efforts

- Sub-($\bar{K}N$)-threshold $\Lambda(1405)$ resonance
- second state $\Lambda(1380)$ predicted from UCHPT
- no direct experimental verification
- confirmed by many critical tests & LQCD