A broad review of the phenomenology of $\gamma\gamma$ interactions

Igor Danilkin

MESON2026
29 June 2026

DFG Deutsche Forschungsgemeinschaft
FOR5327

JOHANNES GUTENBERG UNIVERSITÄT MAINZ

Motivation

- **Photon-photon interactions provide a clean probe of the hadronic systems**
  They allow us to study

$$\gamma^{(*)}\gamma^{(*)} \to \text{hadrons}$$

  in a way that is complementary to hadron-hadron scattering

- Main physics goals:

  Hadron spectroscopy
  scalar, tensor, and axial resonances

  Precision QCD observables
  HLbL input for $(g-2)_\mu$, pion polarizabilities, nucleon (generalized) polarizabilities

  Experimental tools
  realistic simulations for two-photon production at e^+e^- facilities

  Search for ALP and BSM particles in MeV - multi GeV range

Motivation

- **Why dispersion relations?**
  Data are measured in one kinematic region, but amplitudes are needed somewhere else

- Physical two-photon data

$$\gamma\gamma \to \pi\pi,\ \pi\eta,\ K\bar{K},\ \ldots$$

- Need analytic continuation / off-shell extension for:

$$\begin{array}{lll} \pi \text{ polariz :} & s = 0, \quad t = m_\pi^2 & \\ \text{HLbL \& } (g-2) & \gamma^*\gamma^* \to \text{mesons} & \text{(spacelike virtualities)} \\ \text{VCS :} & t \text{ channel : } \gamma^*\gamma \to \pi\pi \to N\bar{N} & \\ \phi \to \gamma\pi^0\pi^0 : & \text{related to } \gamma^*\gamma \to \pi^0\pi^0 & \end{array}$$

- Need amplitudes consistent with

$$\text{analyticity} + \text{unitarity} + \text{crossing}$$

- Motivation
- Formalism for $\gamma^*\gamma^* \to M_1 M_2$:
  Partial waves and kinematic constraints
  Unitarity and Muskhelishvili-Omnès representation
  Left-hand cuts
- Applications:
  Hadronic light-by-light scattering and $(g-2)_\mu$
  Monte Carlo tools
  Virtual Compton scattering off the proton
  Radiative ϕ decays
- Summary

Partial waves and kinematic constraints

- Basic process and the main coupled-channel systems

$$\gamma^*(q_1)\gamma^*(q_2) \to M_1(p_1)M_2(p_2)$$

$$s = (q_1 + q_2)^2, \qquad Q_i^2 = -q_i^2$$

$$I = 0: \quad \pi\pi/K\bar{K}, \qquad I = 1: \quad \pi\eta/K\bar{K}$$

- Start from p.w. helicity amplitudes

$$H_{\lambda_1\lambda_2}(s,\theta) \equiv \epsilon_{1,\mu}\,\epsilon_{2,\nu}\,H^{\mu\nu} = \sum_J (2J+1)\,h^{(J)}_{\lambda_1\lambda_2}(s)\,d^J_{\lambda_1-\lambda_2,0}(\theta)$$

$h^{(J)}_{\lambda_1\lambda_2}$ contain kinematic constraints

Need amplitudes **free of kinematic singularities/constraints** (only possible for the Born-subtracted amplitudes [Bardeen:1969aw, Tarrach:1975tu, Drechsel:1997xv, Colangelo:2015ama])

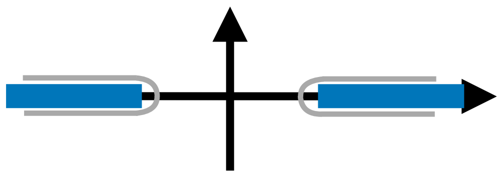
$$\bar{h}_i^{(J)} \equiv h_i^{(J)} - h_i^{(J),\text{Born}}$$

Only these amplitudes satisfy standard dispersion relations [Danilkin:2019opj]

Dispersion relations and unitarity

- After **Born subtraction** and removal of kinematic constraints

$$\bar{h}_i^{(J)}(s) = \int_L \frac{ds'}{\pi} \frac{\text{Im}\, \bar{h}_i^{(J)}(s')}{s'-s} + \int_R \frac{ds'}{\pi} \frac{\text{Im}\, h_i^{(J)}(s')}{s'-s}$$



- Left-hand cut:

$$\text{Im}\, \bar{h}_i^{(J)} \Rightarrow \text{ heavier exchanges } \rho, \omega, K^*, \ldots$$

- Right-hand cut:

$$\text{Im}\, h_i^{(J)}(s) = t^{(J)*}(s)\, \rho(s)\, h_i^{(J)}(s) \quad \text{(unitarity)}$$

- Strategy:

	Omnès matrix	applications
$t^{(J)}(s)$ $\Rightarrow$	$\Omega^{(J)}(s)$ $\Rightarrow$	$\gamma\gamma \to \pi\pi,\ \pi\eta,\ K\bar{K},\ \ldots$

Omnès matrix $\Omega^{(J)}(s)$ is constructed from the hadronic scattering matrix $t^{(J)}(s)$, but only contains right-hand cuts. Details of our construction: [Danilkin:2020pak]

Mushkhelishvili-Omnès representation

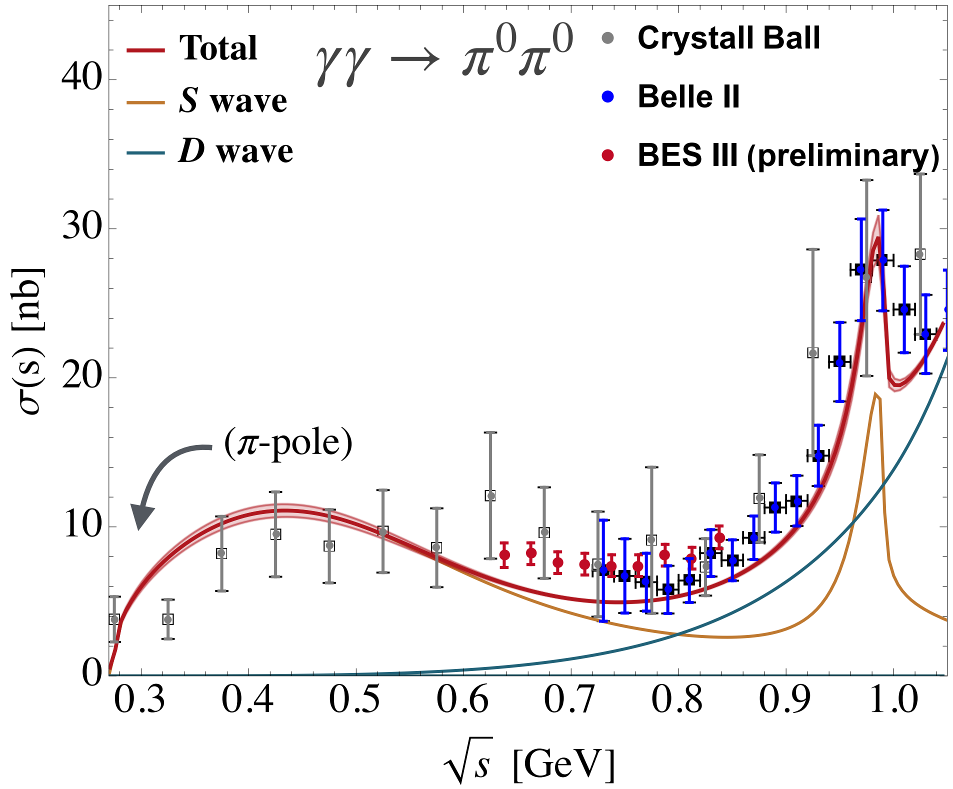
	Omnès matrix	applications
$t^{(J)}(s)$ $\Rightarrow$	$\Omega^{(J)}(s)$ $\Rightarrow$	$\gamma\gamma \to \pi\pi,\ \pi\eta,\ K\bar{K},\ \ldots$

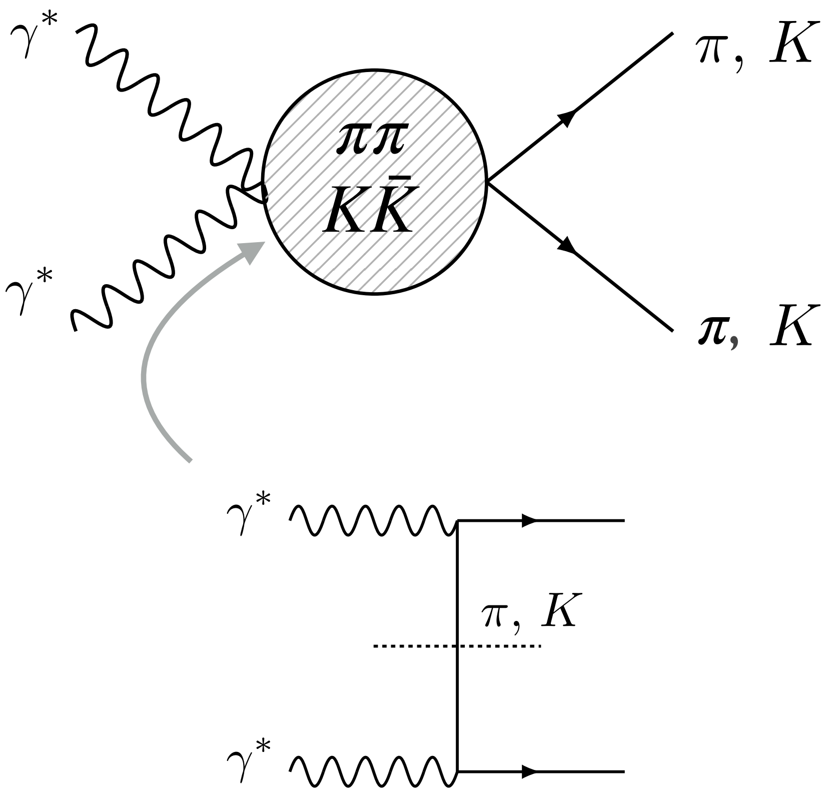
- Muskhelishvili-Omnès representation schematically given by

$$\bar{h}_i^{(J)}(s) = \Omega^{(J)}(s)\left[P_i^{(J)}(s) + \text{left hand cut integrals}\right]$$

The Omnès matrix contains the **universal hadronic rescattering**.
The polynomial/left-hand cut parts contain the **process-dependent production mechanism**

$\gamma\gamma \to \pi\pi/K\bar{K}$

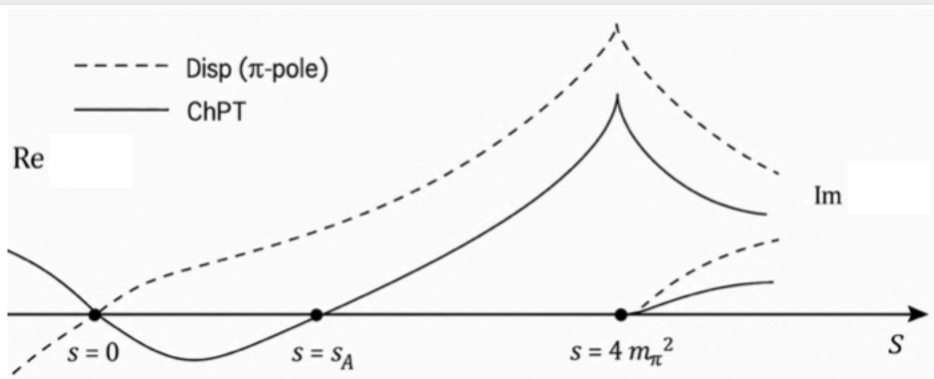




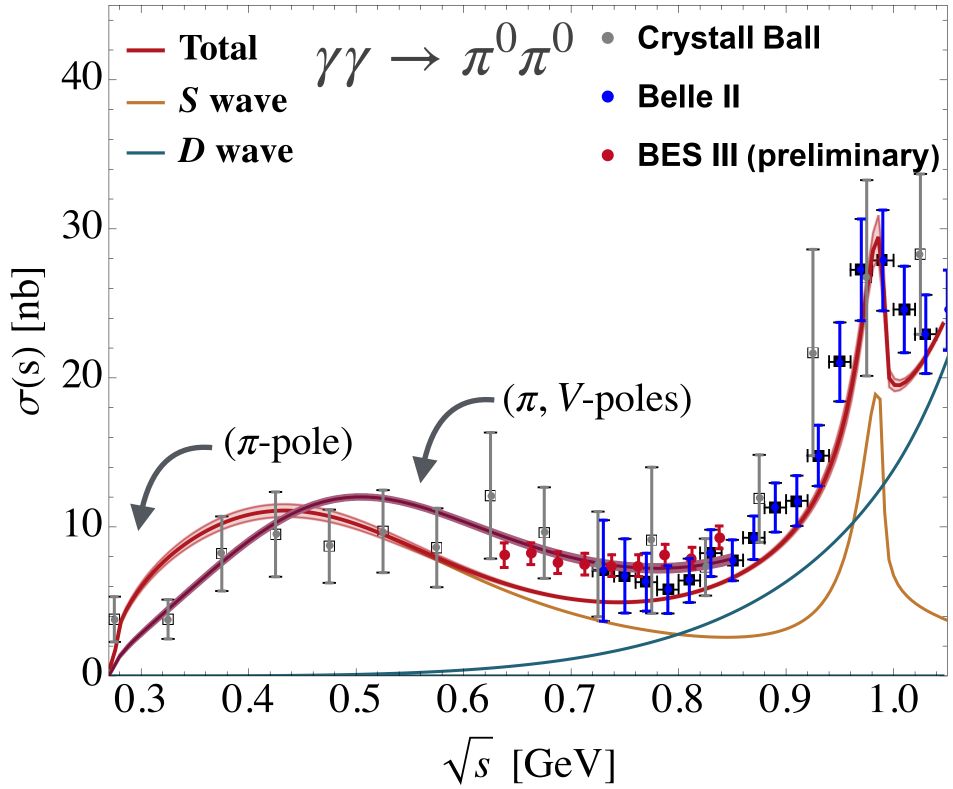
Unsubtracted dispersion relation for s-wave [Colangelo:2017fiz, Danilkin:2018qfn]

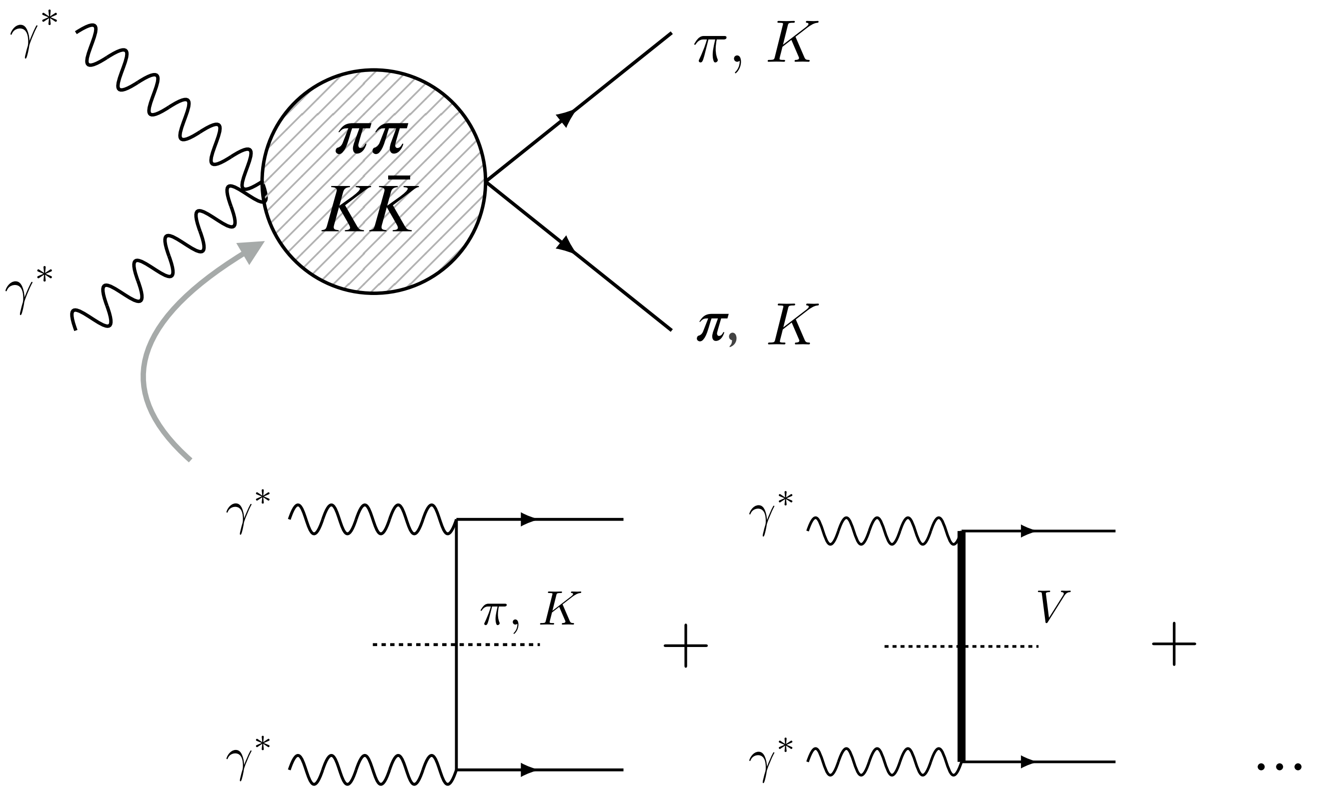
- Left-hand cuts: π/K poles
- $\Gamma_{\gamma\gamma}(f_0(500), f_0(980))$ consistent with other analyses (e.g. [Dai:2014zta])
- $(\alpha_1 - \beta_1)_{\pi^\pm}$ consistent with χPT
- Neutral channel tension $(\alpha_1 - \beta_1)_{\pi^0} \sim +9 \times 10^{-4}\,\mathrm{fm}^3$, $(\alpha_1 - \beta_1)^{\chi\mathrm{PT}}_{\pi^0} = -1.9(2) \times 10^{-4}\,\mathrm{fm}^3$

$$\bar{h}_i^{(J)}(s) \propto (\alpha_1 - \beta_1)_\pi\, s + \ldots$$



$\gamma\gamma \to \pi\pi/K\bar{K}$





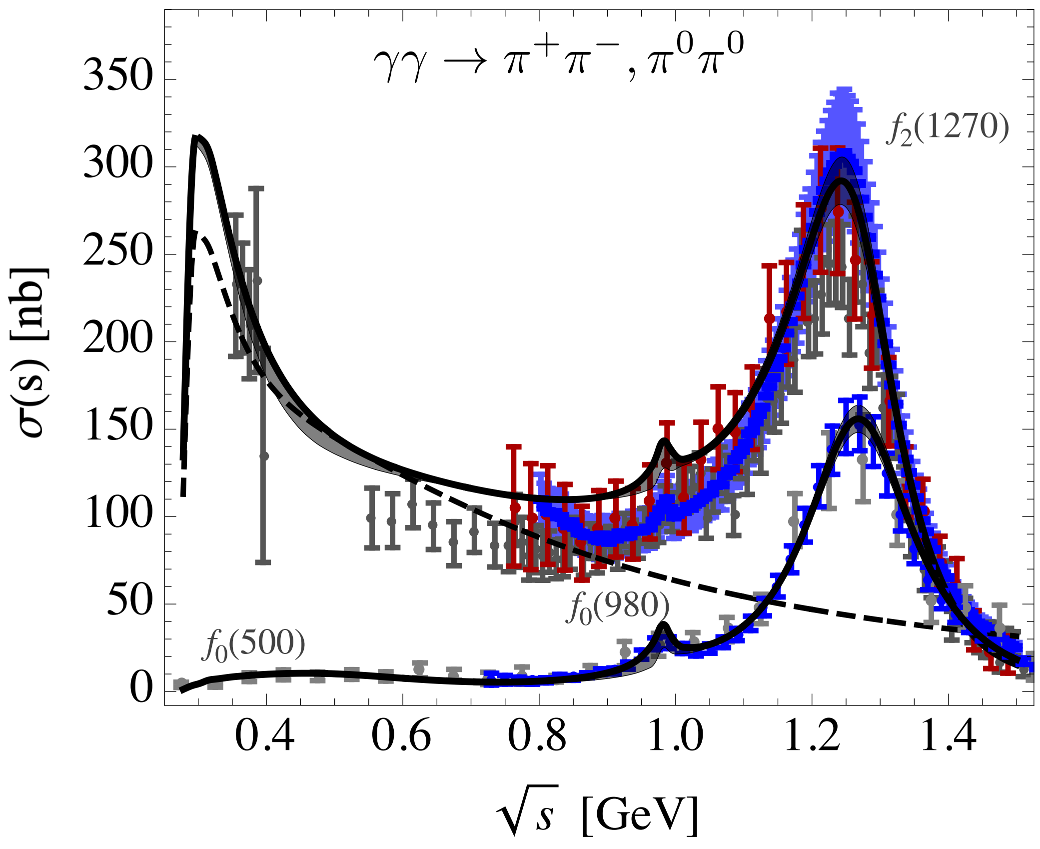
Unsubtracted dispersion relation for s-wave [Colangelo:2017fiz, Danilkin:2018qfn]

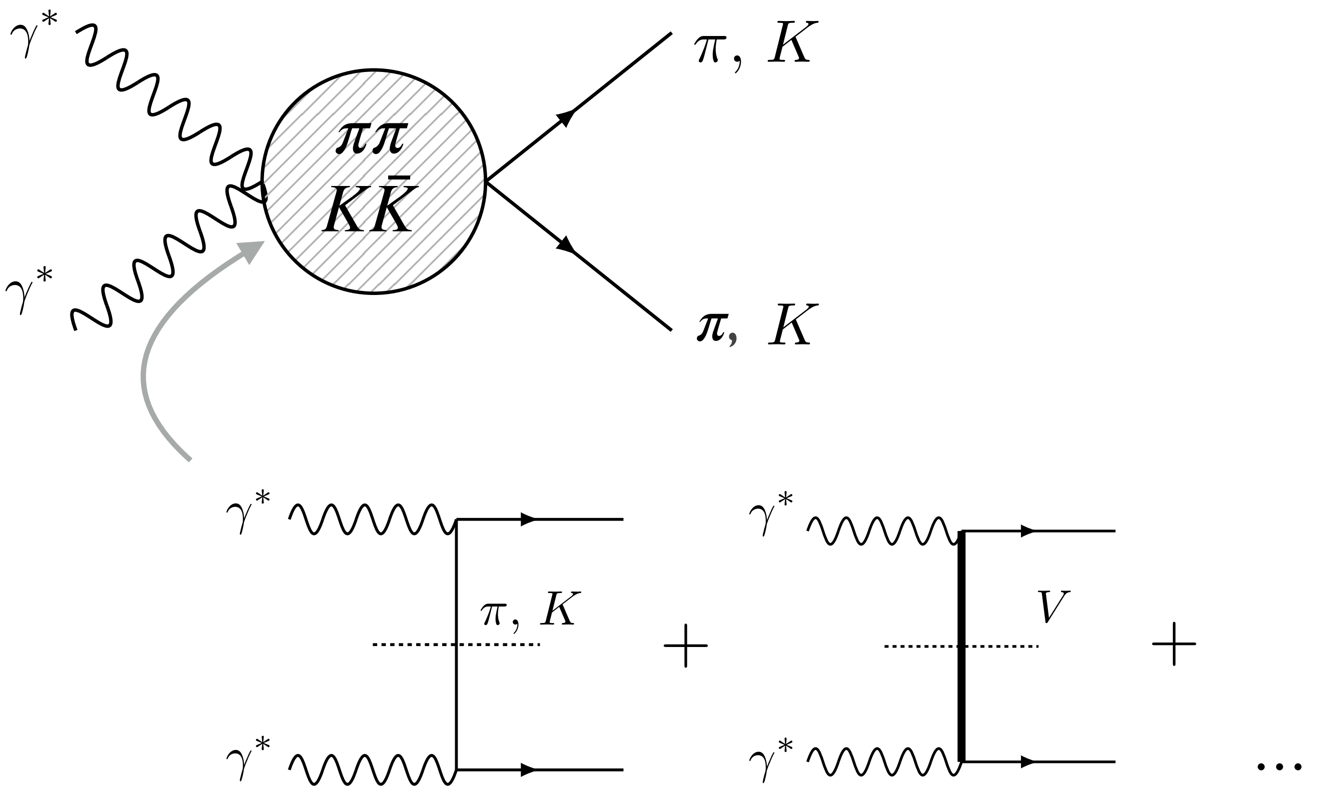
- Left-hand cuts: π/K poles
- $\Gamma_{\gamma\gamma}(f_0(500), f_0(980))$ consistent with other analyses (e.g. [Dai:2014zta])
- $(\alpha_1 - \beta_1)_{\pi^\pm}$ consistent with χPT
- Neutral channel tension $(\alpha_1 - \beta_1)_{\pi^0} \sim +9 \times 10^{-4}\,\text{fm}^3$, $(\alpha_1 - \beta_1)^{\chi PT}_{\pi^0} = -1.9(2) \times 10^{-4}\,\text{fm}^3$

Subtracted dispersion relation for s-wave

- Left-hand cuts: $\pi/K + V$ poles
- Accurate fit to $\gamma\gamma \to \pi^0\pi^0$ via subtraction constants
- Cure $(\alpha_1 - \beta_1)_{\pi^0}$ by including **Adler zero** [Dai:2016ytz, Ermolina:2024daf]

$\gamma\gamma \to \pi\pi/K\bar{K}$





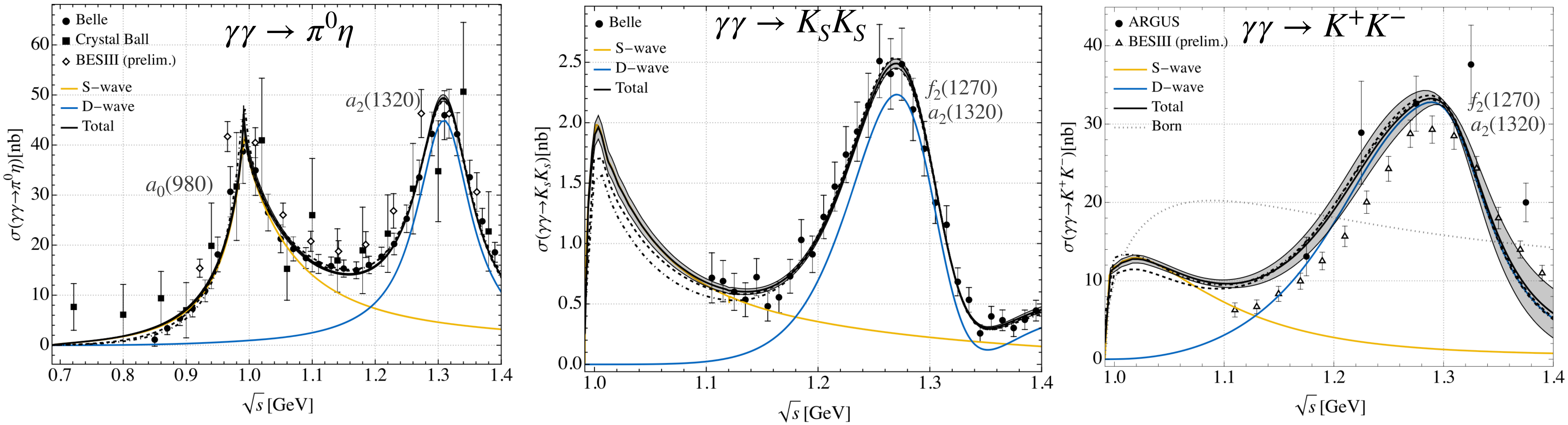
Unsubtracted dispersion relation for d-wave

- Heavier left-hand cuts are essential [Garcia-Martin:2010kyn]
- Approximate heavier cuts by V-poles $C_{\rho\pi\gamma} \simeq C_{\omega\pi\gamma}/3 \approx g^{\text{eff}}_{VP\gamma} = 0.33\,\text{GeV}^{-1}$
  $g^{\text{PDG}}_{VP\gamma} = 0.37(2)\,\text{GeV}^{-1}$ [Danilkin:2018qfn]

Data used: $\pi\pi/K\bar{K}$ scattering data (Roy-like analyses)
$\gamma\gamma \to \pi^0\pi^0$ used to justify left-hand cut approximation for s-wave
and mildly tune left-hand cuts for d-wave

Prediction for $\gamma^*\gamma^* \to \pi\pi/K\bar{K}_{I=0}$ [Danilkin:2019opj, Hoferichter:2019nlq]

$\gamma\gamma \to \pi\eta/K\bar{K}$



Challenge: ❗ no direct $\pi\eta/K\bar{K}$ scattering data

$\gamma\gamma \to \pi\eta/K\bar{K}$ data used to constrain $\pi\eta/K\bar{K}$ amplitude [Deineka:2024mzt]

(see also [Lu:2020qeo])

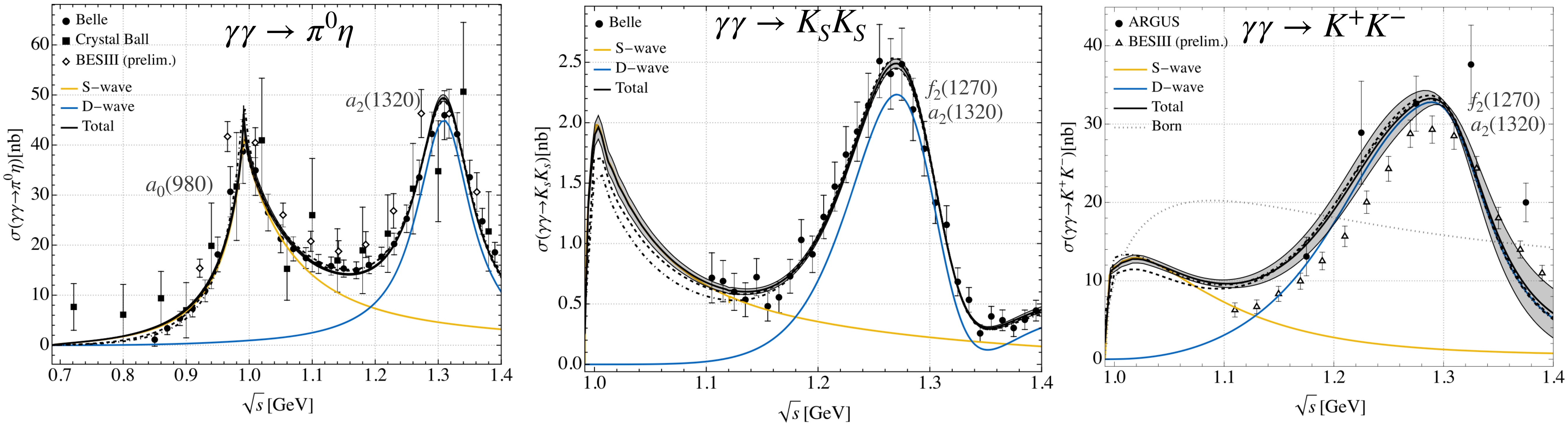
Unsubtracted dispersion relation for s-wave

- Left-hand cuts: K pole

Subtracted dispersion relation for s-wave

- Left-hand cuts: $K + V$ poles

$\gamma\gamma \to \pi\eta/K\bar{K}$



Challenge: **!** no direct $\pi\eta/K\bar{K}$ scattering data

$\gamma\gamma \to \pi\eta/K\bar{K}$ data used to constrain $\pi\eta/K\bar{K}$ amplitude [Deineka:2024mzt]

(see also [Lu:2020qeo])

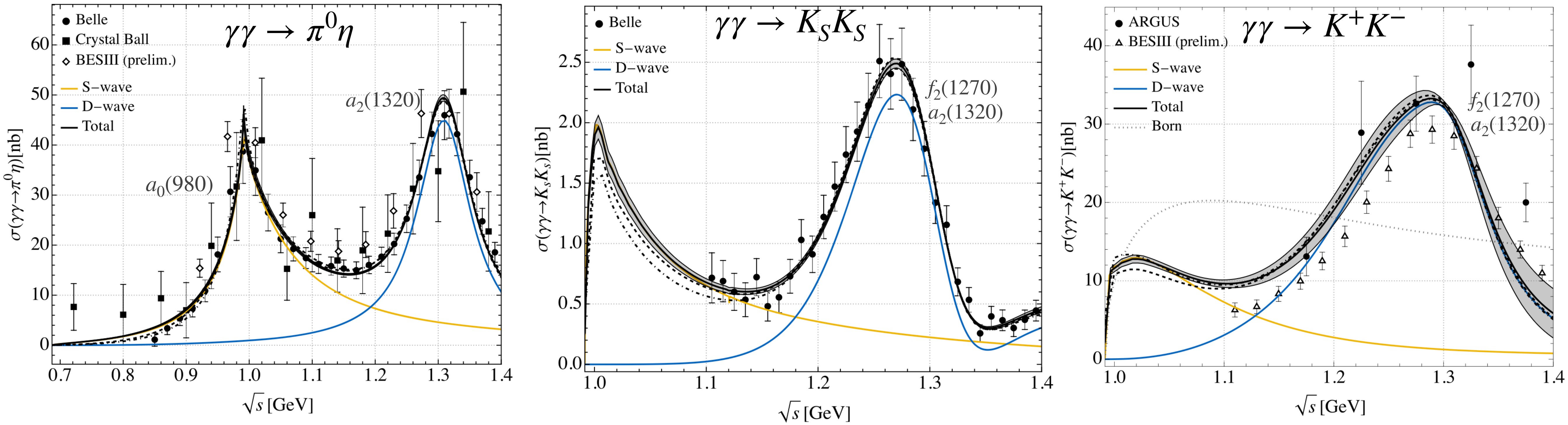
Unsubtracted dispersion relation for s-wave

- Left-hand cuts: K pole
- no Adler zero $\gamma\gamma \to \pi^0\eta$
- Prediction for $\gamma^*\gamma^* \to \pi\eta/K\bar{K}_{I=1}$

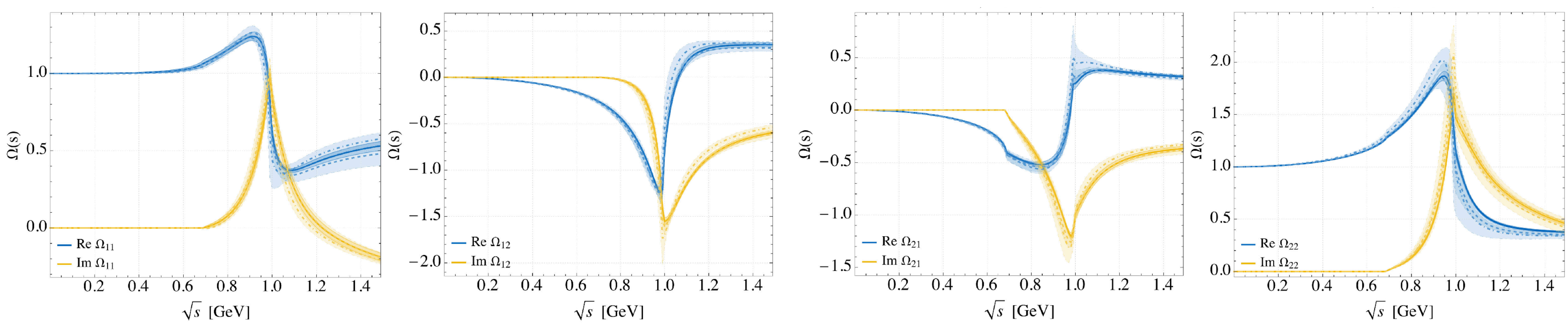
Subtracted dispersion relation for s-wave

- Left-hand cuts: $K + V$ poles
- Adler zero $\gamma\gamma \to \pi^0\eta$

$\gamma\gamma \to \pi\eta/K\bar{K}$



- **Unsubtracted** and **Subtracted** dispersion relations for $\gamma\gamma \to \pi\eta/K\bar{K}$ yield very similar hadronic $\pi\eta/K\bar{K}$ amplitudes



- Pole parameters of $a_0(980)$ are carefully extracted [Deineka:2024mzt]

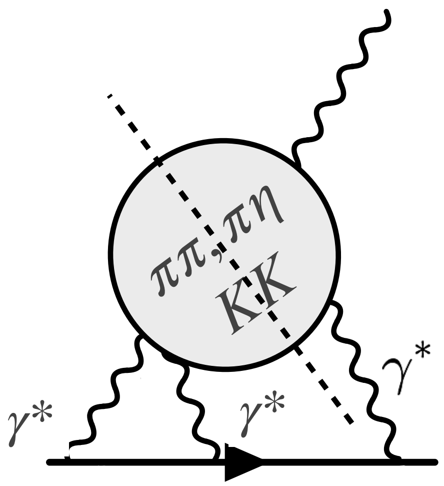
- Motivation

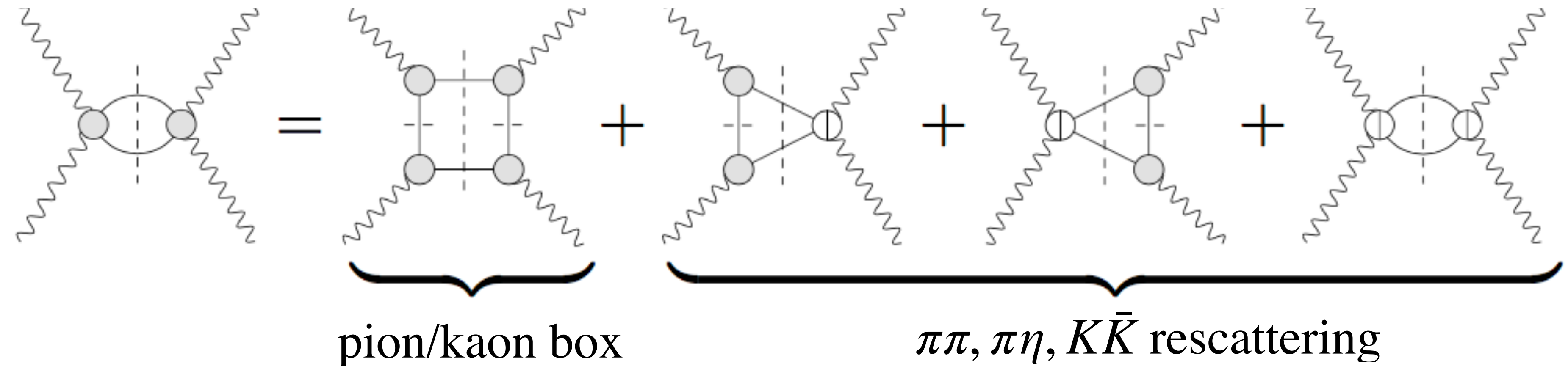
- Formalism for $\gamma^*\gamma^* \to M_1 M_2$:
  Partial waves and kinematic constraints
  Unitarity and Muskhelishvili-Omnès representation
  Left-hand cuts

- Applications:
  Hadronic light-by-light scattering and $(g-2)_\mu$
  Monte Carlo tools
  Virtual Compton scattering off the proton
  Radiative ϕ decays

- Summary

Hadronic light-by-light scattering and $(g-2)_\mu$



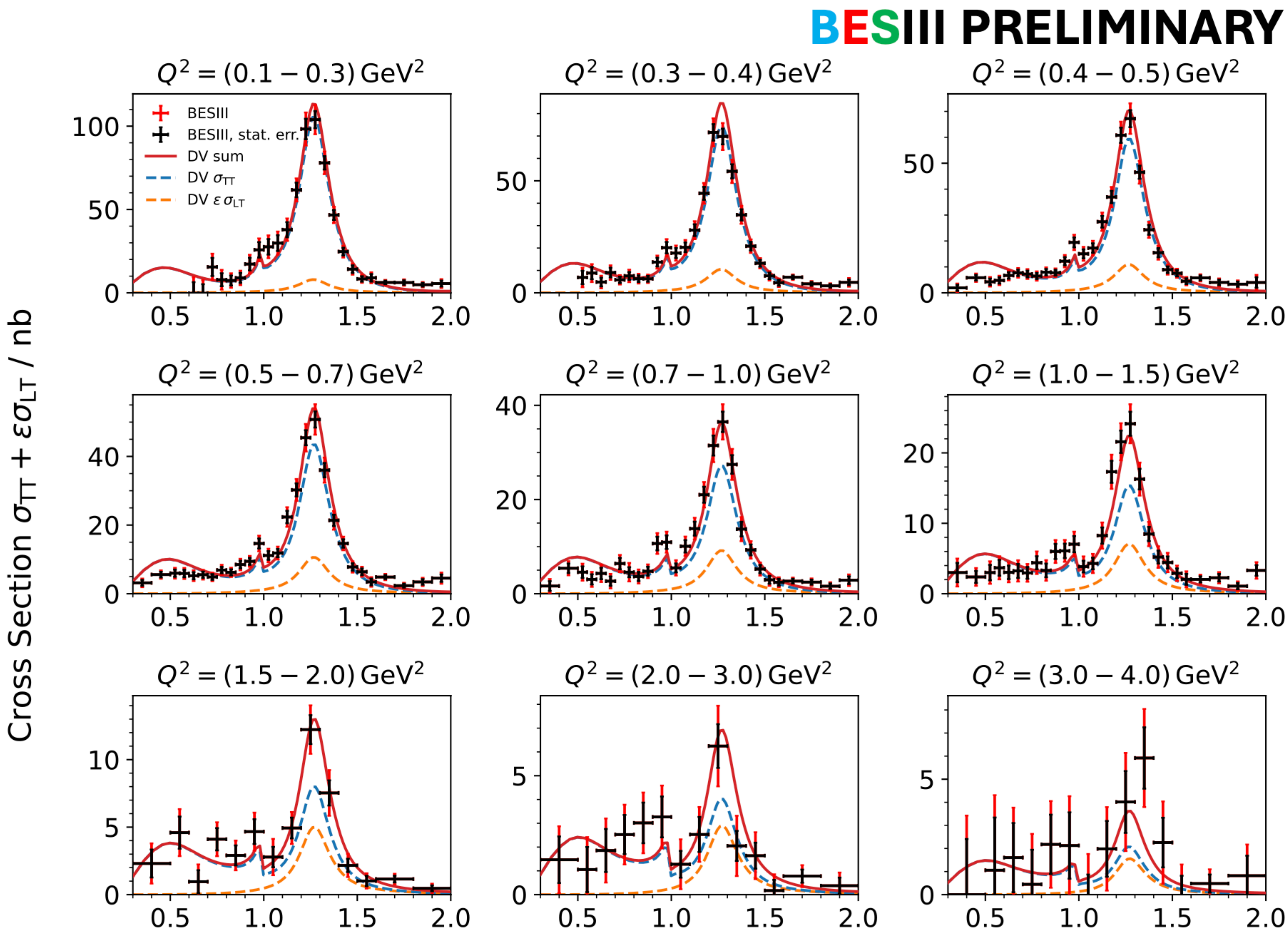


$$a_\mu^{\text{HLbL}} = \frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \sqrt{1-\tau^2} Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1, Q_2, Q_3)\, \bar{\Pi}_i(Q_1, Q_2, Q_3)$$

$a_\mu[\text{box}]_{\pi\pi,K\bar{K}} = -16.4(2) \times 10^{-11}$	Born	[Colangelo:2017fiz]
$a_\mu[\text{S-wave}, I=0]_{\pi\pi} = -9.3(9) \times 10^{-11}$	$f_0(500)$	[Colangelo:2017fiz]
$a_\mu[\text{S-wave}, I=0]_{\pi\pi,K\bar{K}} = -9.8(1.0) \times 10^{-11}$	$f_0(500)/f_0(980)$	[Danilkin:2021icn]
$a_\mu[\text{S-wave}, I=1]_{\pi\eta,K\bar{K}} = -0.44(5) \times 10^{-11}$	$a_0(980)$	[Deineka:2024mzt]

- These results enter prominently in the $(g-2)_\mu$ White Paper [Aliberti:2025beg]
- Upcoming BESIII data for $\gamma\gamma^* \to \pi\pi, \pi^0\eta$ ($Q^2 = 0.2 - 2.0\,\text{GeV}^2$)

Hadronic light-by-light scattering and $(g-2)_\mu$



[Max Lellmann PhiPsi ‘26 Pisa]
https://indico.sns.it/event/140/contributions/1259/attachments/433/1230/260610_PhiPsi.pdf

- Motivation

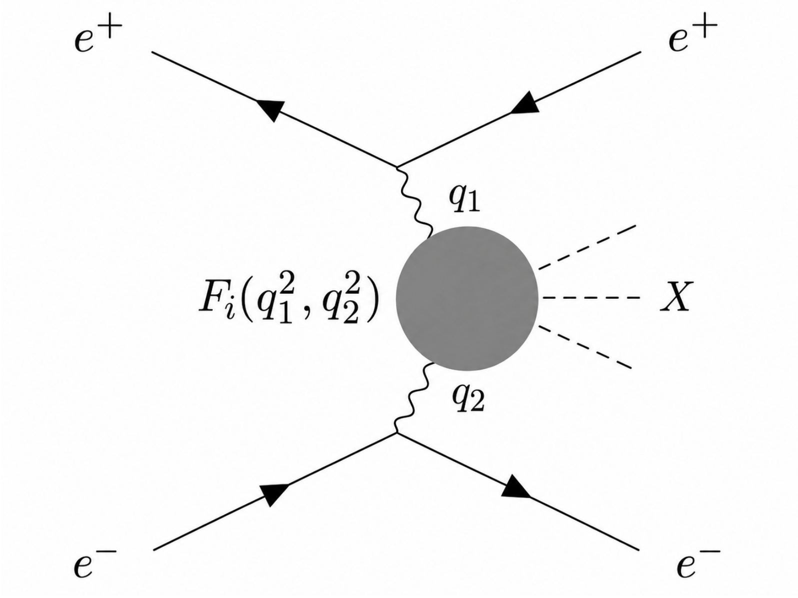
- Formalism for $\gamma^*\gamma^* \to M_1 M_2$:
  Partial waves and kinematic constraints
  Unitarity and Muskhelishvili-Omnès representation
  Left-hand cuts

- Applications:
  Hadronic light-by-light scattering and $(g-2)_\mu$
  Monte Carlo tools
  Virtual Compton scattering off the proton
  Radiative ϕ decays

- Summary

Monte Carlo tools

- To access two-photon couplings/TFFs of pseudo-scalar, scalar, axial, tensor mesons we developed a new MC generator for $e^+e^- \to e^+e^-X$ [Lellmann:2025aje]



exact QED production
+
hadronic amplitudes
+
flexible phase space

HadroTOPS = event generator for Hadron production in Two-Photon Scattering

- Currently it includes

pseudoscalars (π^0, η, η')
two-meson final states ($\pi\pi$, $\pi\eta$, ...)
three-meson final states ($\pi\pi\eta$, ...)

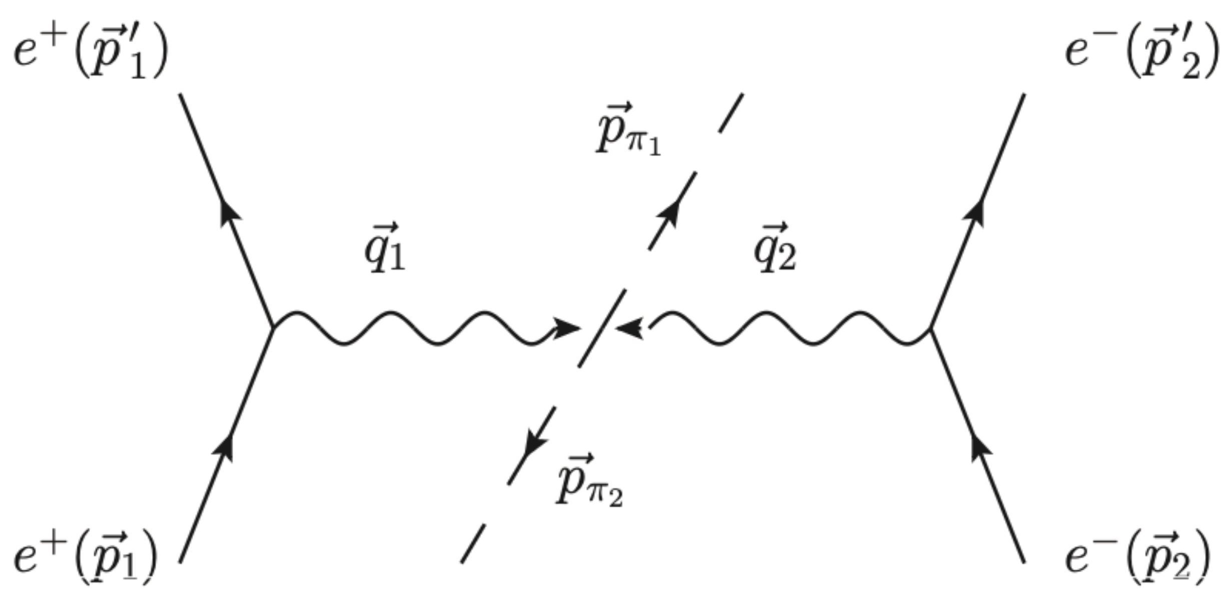
$\left.\right\}$ $f_0(500), f_0(980), a_0(980)$ $f_2(1270), a_2(1320), f_1(1285)$

Monte Carlo tools

- MC generator is needed for

  acceptance / normalization / single-tag and double-tag studies

- **HadroTOPS** extends the classic $\gamma^*\gamma^*$ response functions formalism [Bonneau:1973kg, Budnev:1974de] from inclusive observables to fully exclusive $e^+e^- \to e^+e^-\pi\pi$.
  The **full azimuthal-angle dependence** is retained, giving access to interference response functions



$$\frac{d\sigma_0}{d\cos\theta_\pi} \propto \left|H_{++}\right|^2$$
$$\frac{d\sigma_2}{d\cos\theta_\pi} \propto \left|H_{+-}\right|^2$$
$$\frac{d\tau_{T2}}{d\cos\theta_\pi} \propto \mathrm{Re}\left(H^*_{++}H_{+-}\right)$$

- For $\pi\pi$, **HadroTOPS** results are consistent with independent **Ekhara 3.2** [Henryk Czyż]

- Motivation

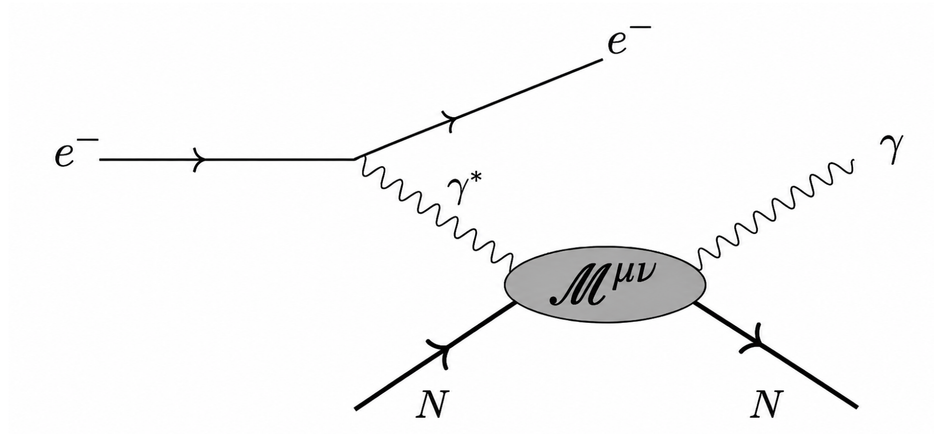
- Formalism for $\gamma^*\gamma^* \to M_1 M_2$:
  Partial waves and kinematic constraints
  Unitarity and Muskhelishvili-Omnès representation
  Left-hand cuts

- Applications:
  Hadronic light-by-light scattering and $(g-2)_\mu$
  Monte Carlo tools
  Virtual Compton scattering off the proton
  Radiative ϕ decays

- Summary

Virtual Compton scattering off the proton

- Experimentally, in $e^- p \to e^- \gamma p$ one has Bethe-Heitler + Born VCS + non-Born VCS.
  The non-Born VCS amplitudes give access to the generalized polarizabilities



$$\alpha_{E1}(Q^2),\ \beta_{M1}(Q^2)$$

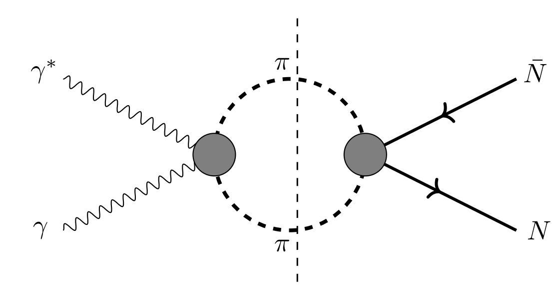
- The tensor $\mathcal{M}^{\mu\nu}$ is decomposed in a gauge-invariant basis $\mathcal{M}^{\mu\nu} = \sum_{i=1}^{12} F_i(Q^2, \nu, t)\, L_i^{\mu\nu}$

Fixed-t dispersion relations for F_i
Unsubtracted DRs require input from electro- and photo-production data [Pasquini:2001yy]

$$s\text{-channel:} \quad \gamma^* p \to \pi N \to \gamma p$$

Some F_i require once-subtracted DRs [Drechsel:1999rf, Danilkin:2026wkl]
We write dispersion relations for the subtraction functions $F_i^{\mathrm{NB}}(Q^2, \nu = 0, t)$

$$t\text{-channel RHC:} \quad \gamma^* \gamma \to \pi\pi \to N\bar{N}$$
$$t\text{-channel LHC:} \quad \Delta(1232) \text{ exchange}$$

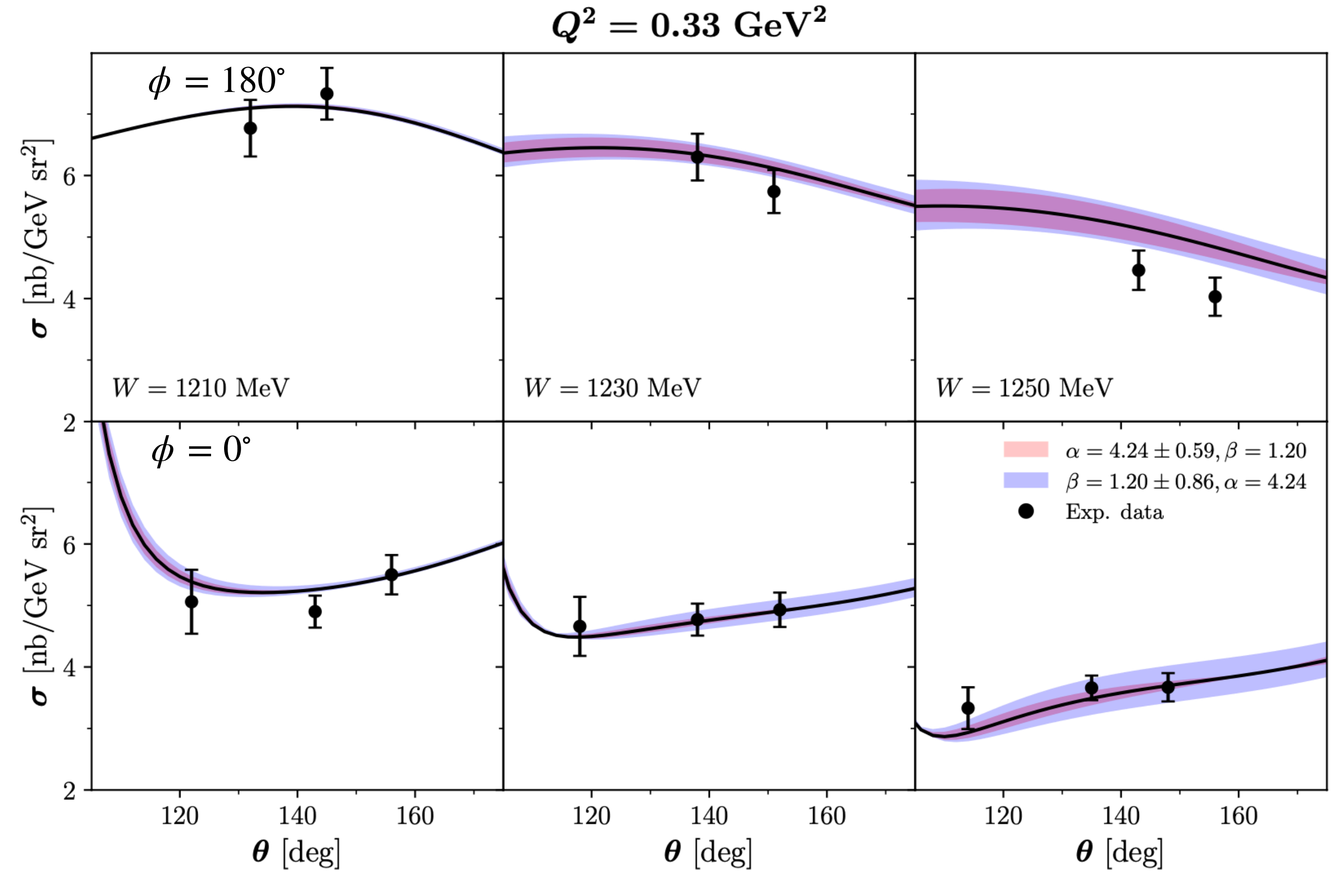


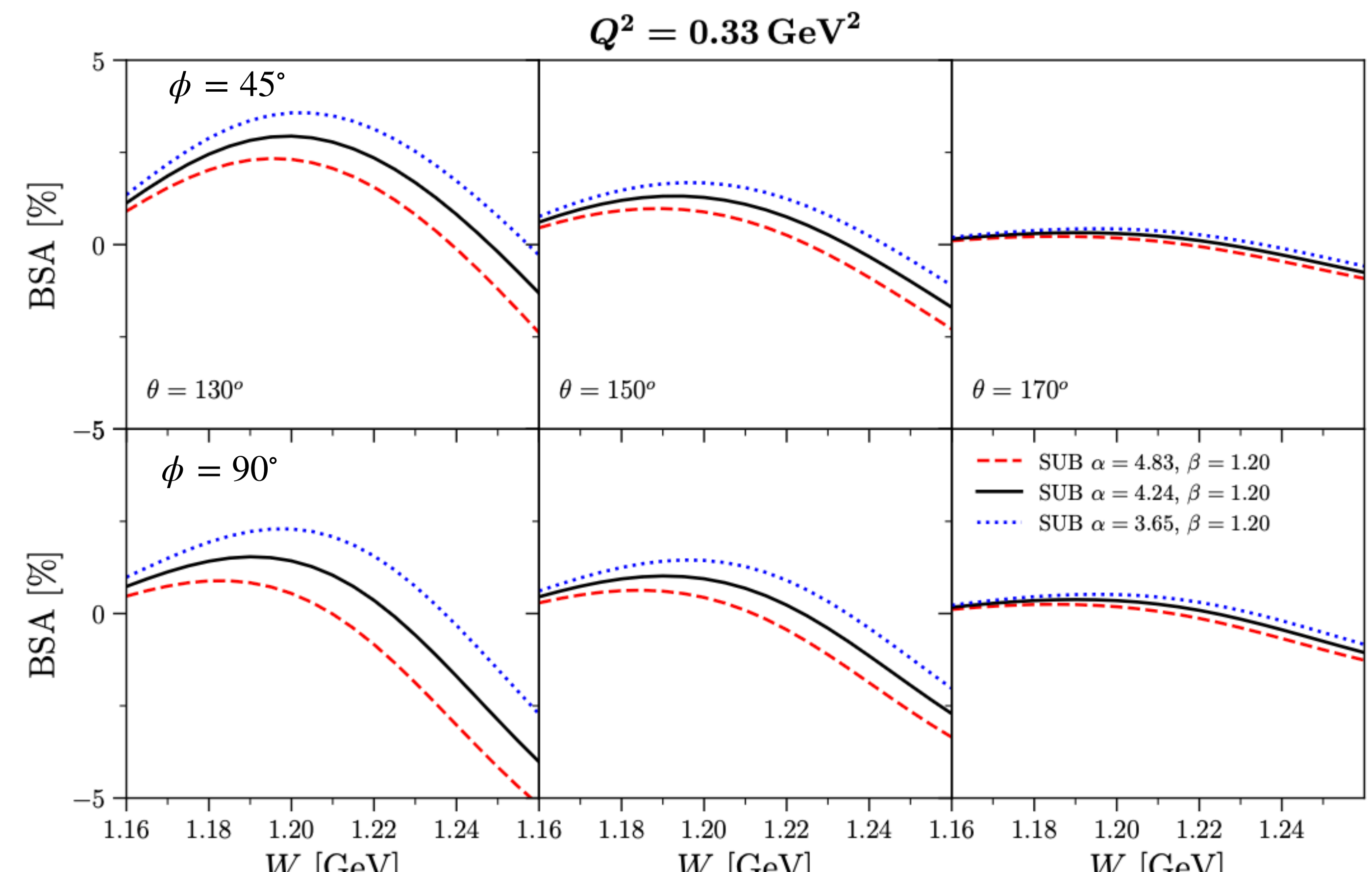
Virtual Compton scattering off the proton

- Good general agreement with data [Li:2022sqg] without performing the fit

- For t-channel, we used input from $\gamma^*\gamma \to \pi\pi$ [Danilkin:2018qfn] $\pi\pi \to N\bar{N}$ [Hoferichter:2015hva]

- The beam-spin asymmetry has strong sensitivity to $\alpha_{E1}(Q^2)$. The subtracted DR formalism will allow a new fit to constrain $\alpha_{E1}(Q^2), \beta_{M1}(Q^2)$ in the upcoming VCS-IIp and VCS-IIIp experiments [VCS-II:2023jhu]





- Motivation
- Formalism for $\gamma^*\gamma^* \to M_1 M_2$:
  Partial waves and kinematic constraints
  Unitarity and Muskhelishvili-Omnès representation
  Left-hand cuts
- Applications:
  Hadronic light-by-light scattering and $(g-2)_\mu$
  Monte Carlo tools
  Virtual Compton scattering off the proton
  Radiative ϕ decays
- Summary

Radiative ϕ decays

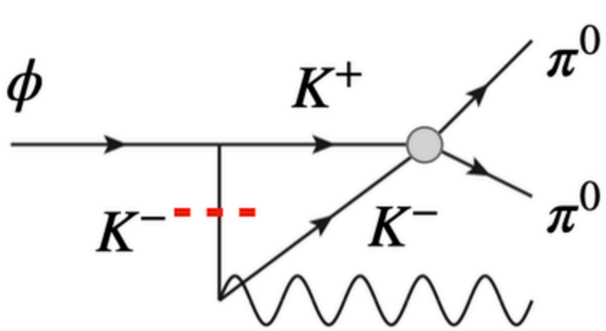
- The radiative decays $\phi \to \gamma\pi^0\pi^0$ and $\phi \to \gamma\pi^0\eta$ were historically proposed as probes of the scalar mesons $f_0(980)$ and $a_0(980)$, respectively.

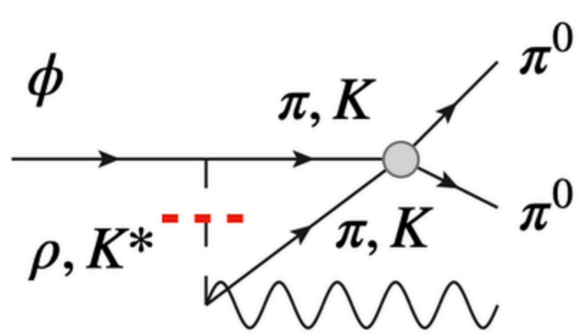
  The key idea is that $\phi \simeq s\bar{s}$ couples strongly to $K\bar{K}$, so these decays are sensitive to scalar dynamics near the $K\bar{K}$ threshold.

- Classic mechanism: kaon-loop model with phenomenological scalar amplitudes [Achasov:2001cj]

$$\begin{aligned}\phi \to K^+K^- &\to \gamma\, f_0(980) \to \gamma\,\pi^0\pi^0 \\ &\to \gamma\, a_0(980) \to \gamma\,\pi^0\eta\end{aligned}$$

- Nowadays, the scalar-isoscalar $\pi\pi/K\bar{K}$ channel is well constrained by Roy-like analyses. Therefore, a dispersive approach **can quantify the kaon-loop dominance hypothesis**, which in dispersive language corresponds to the K-pole left-hand cut



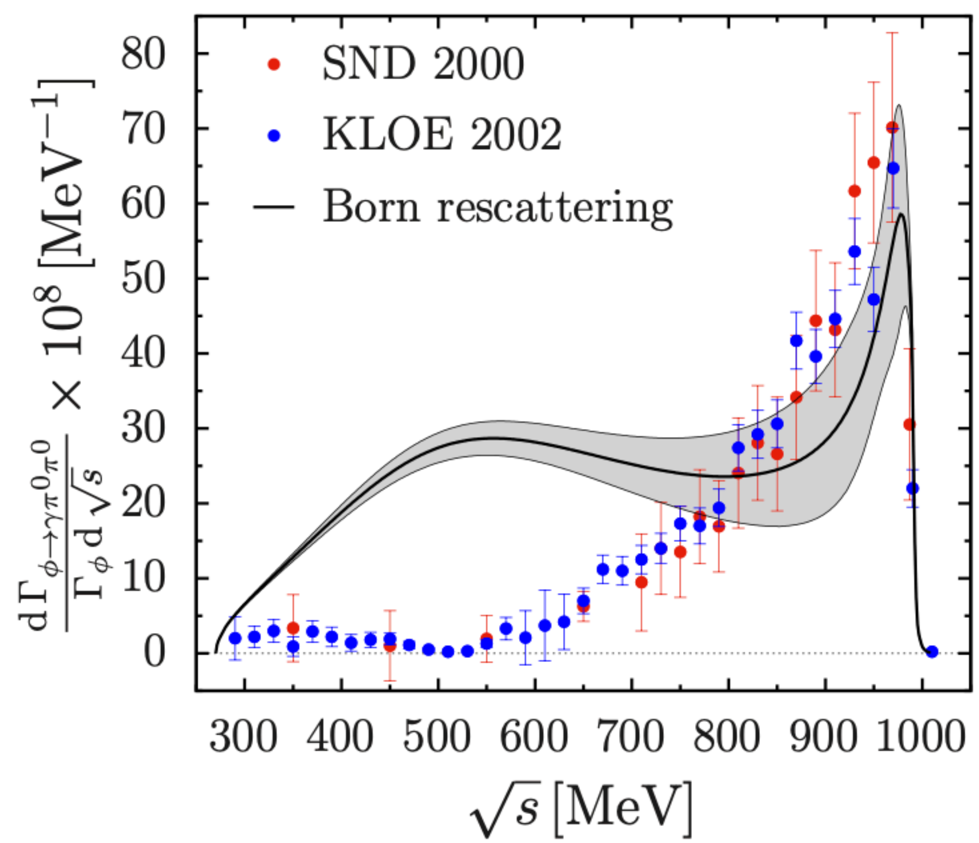


Radiative ϕ decays

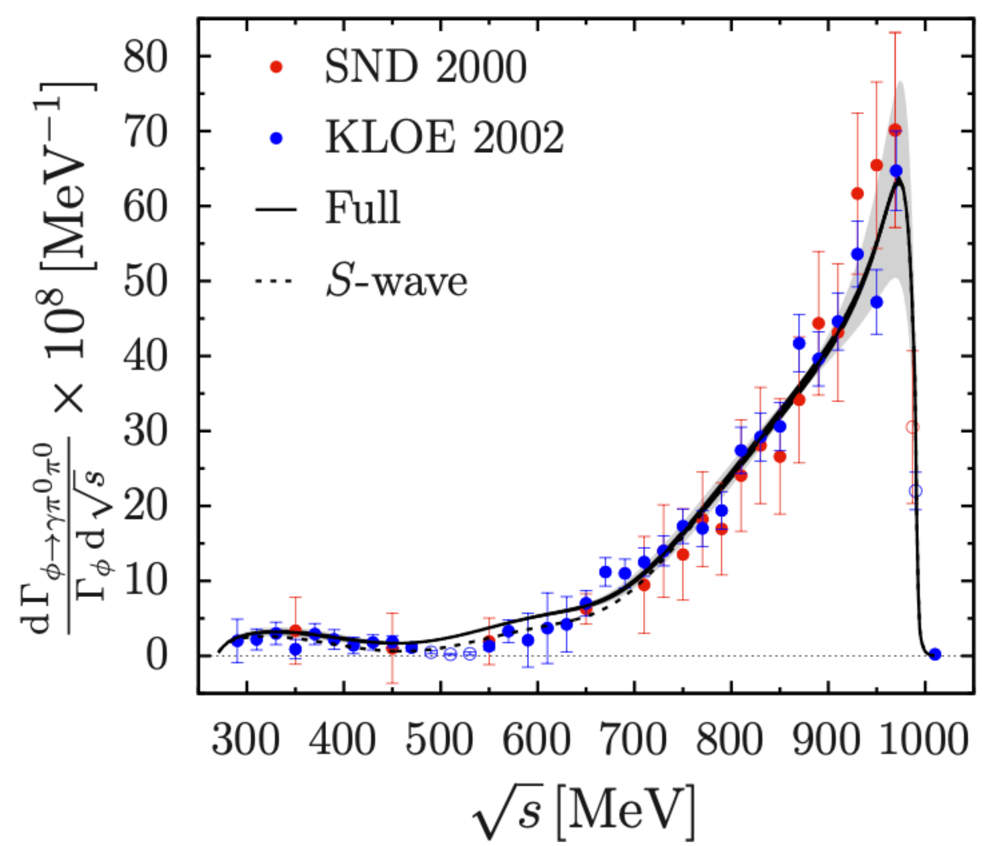
- The decays $\phi \to \gamma\pi^0\pi^0$, $\phi \to \gamma\pi^0\eta$ are also an ideal testing ground for $\gamma\gamma^* \to \pi^0\pi^0$, $\gamma\gamma^* \to \pi^0\eta$, since they probe the same scalar scattering and closely related left-hand cuts. The main challenge is the **decay kinematics**.

- Dispersive program: $\phi \to \gamma\pi^0\eta$ [Moussallam:2021dpk], $\phi \to \gamma\pi^0\pi^0$ [Hoid:2026atr]

Unsubtracted dispersion relation
with K-pole left-hand cuts



Once-subtracted dispersion relation
with K, ρ, K^*-pole left-hand cuts



- Motivation
- Formalism for $\gamma^*\gamma^* \to M_1 M_2$:
  Partial waves and kinematic constraints
  Unitarity and Muskhelishvili-Omnès representation
  Left-hand cuts
- Applications:
  Hadronic light-by-light scattering and $(g-2)_\mu$
  Monte Carlo tools
  Virtual Compton scattering off the proton
  Radiative ϕ decays
- Summary

Summary

- Photon-photon reactions provide clean access to scalar, tensor, and axial meson dynamics.
- Dispersive amplitudes implement

analyticity + unitarity + crossing

and connect

$\gamma\gamma$ data $\leftrightarrow$ resonance dynamics $\leftrightarrow$ precision observables

- Applications discussed:

HLbL in $(g-2)_\mu$,	HadroTOPS,	ϕ radiative decays,	VCS

- **Extra:** we also performed a data-driven dispersive analysis of the heavier channel $\gamma\gamma \to D\bar{D}$ [Deineka:2021aeu] using BaBar and Belle data. We found no evidence for a broad X(3860) pole, but a $D\bar{D}$ bound-state, consistent with earlier molecular-state predictions [Wong:2003xk, Gamermann:2006nm, Dong:2021juy]

Back-up

Mushkhelishvili-Omnès representation

	Omnès matrix	applications
$t^{(J)}(s)$ $\Rightarrow$	$\Omega^{(J)}(s)$ $\Rightarrow$	$\gamma\gamma \to \pi\pi,\ \pi\eta,\ K\bar{K},\ \ldots$

- Muskhelishvili-Omnès representation schematically given by

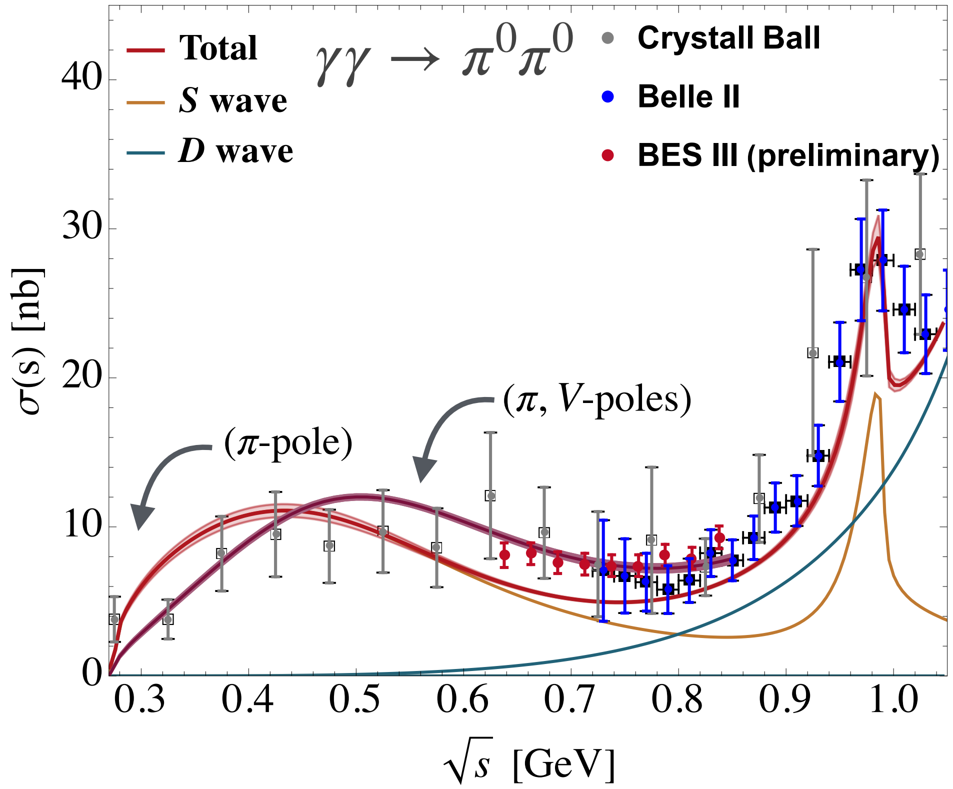
$$\bar{h}_i^{(J)}(s) = \Omega^{(J)}(s)\left[P_i^{(J)}(s) + \text{left hand cut integrals}\right]$$

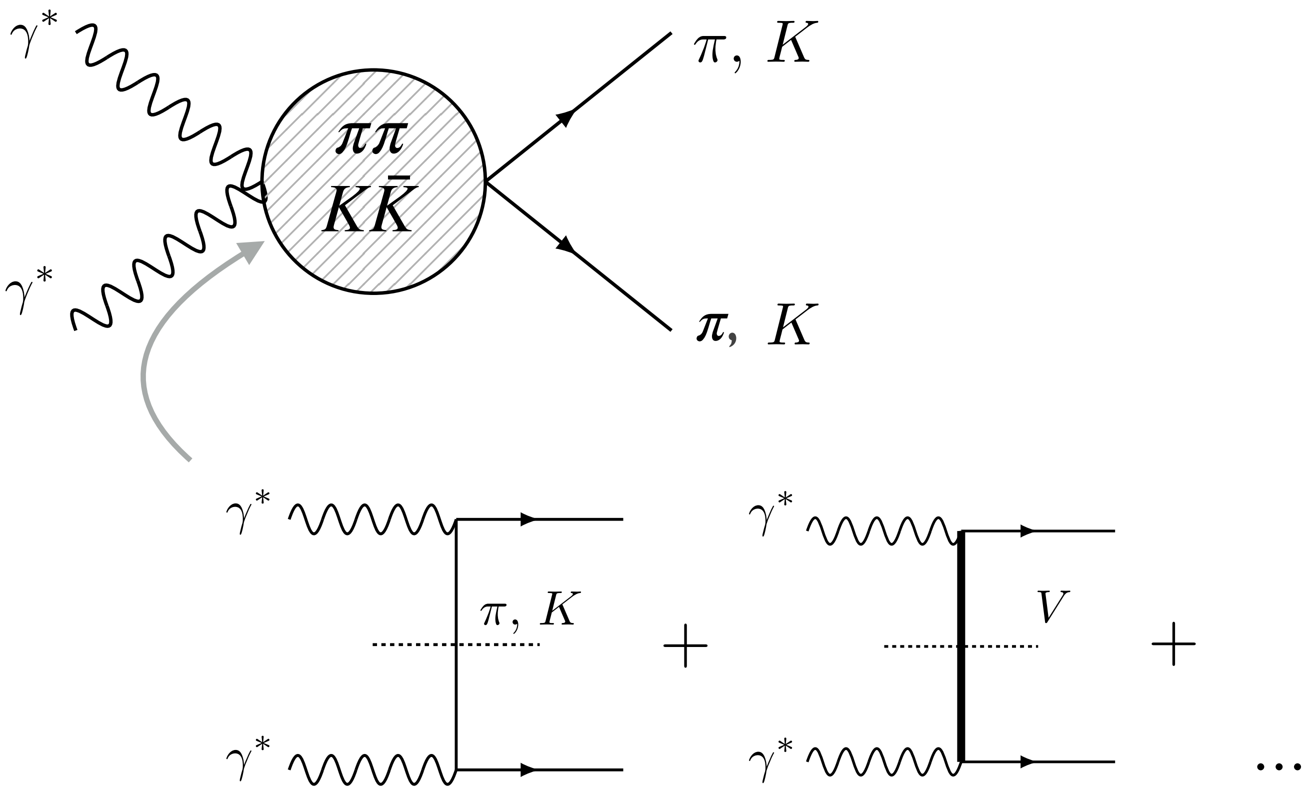
The Omnès matrix contains the **universal hadronic rescattering**.
The polynomial/left-hand cut parts contain the **process-dependent production mechanism**

- Different MO representations can be equivalent once the polynomial ambiguity and asymptotic prescription are fixed [Garcia-Martin:2010kyn, Hoferichter:2019nlq, Moussallam:2021dpk, Hoid:2026atr]

- For $\gamma^*\gamma^* \to \pi\pi, \ldots$ left-hand cuts produces "anomalous thresholds" for large virtualities [Hoferichter:2019nlq, Danilkin:2019opj]

Pion polarizabilities



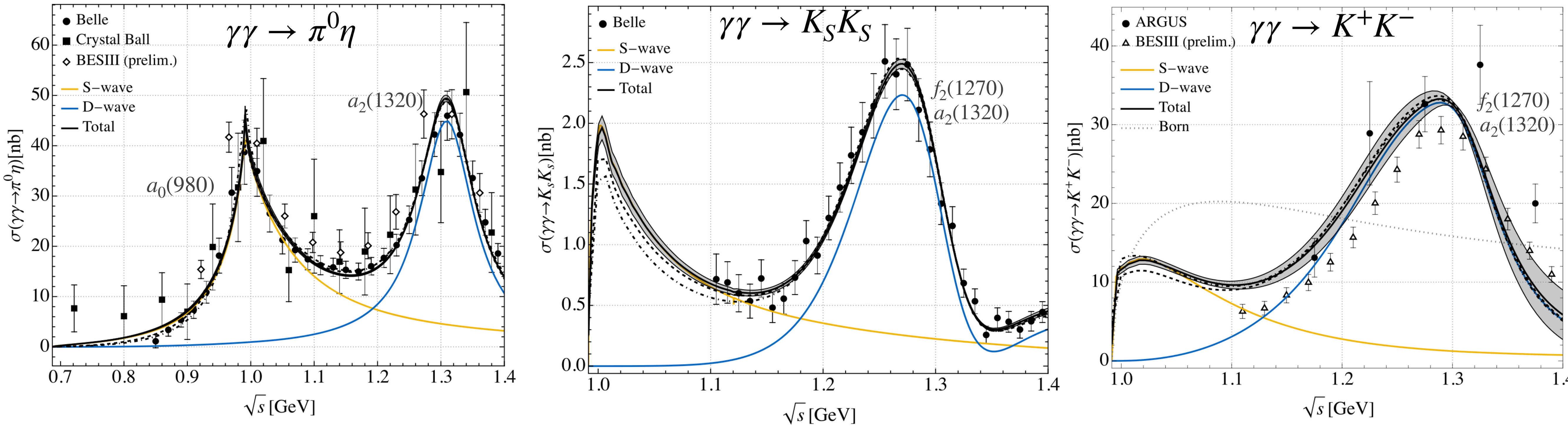


		Dispersive			ChPT	Experiment
		Present work		Garcia-Martin et al.		
		π pole	π, V poles	π, V, A, T pole	NNLO	COMPASS
$(\alpha_1-\beta_1)_{\pi^\pm}$	10^{-4}fm^3	5.1	$2.4\,(4)(3)^{+1.0}_{-0.0}$	4.7	5.7 (1.0)	$4.0\,(1.2)_{\text{stat}}\,(1.4)_{\text{sys}}$
$(\alpha_1-\beta_1)_{\pi^0}$	10^{-4}fm^3	8.4	$-1.3\,(3)(0)^{+0.0}_{-0.3}$	−1.25 (17)	−1.9 (2)	–
$(\alpha_2-\beta_2)_{\pi^\pm}$	10^{-4}fm^5	18.1	$16.5\,(4)(2)^{+2.1}_{-0.0}$	14.7 (2.1)	16.2/21.6	–
$(\alpha_2-\beta_2)_{\pi^0}$	10^{-4}fm^5	24.8	$30.0\,(4)(3)^{+4.3}_{-0.0}$	32.1 (2.1)	37.6 (3.3)	–

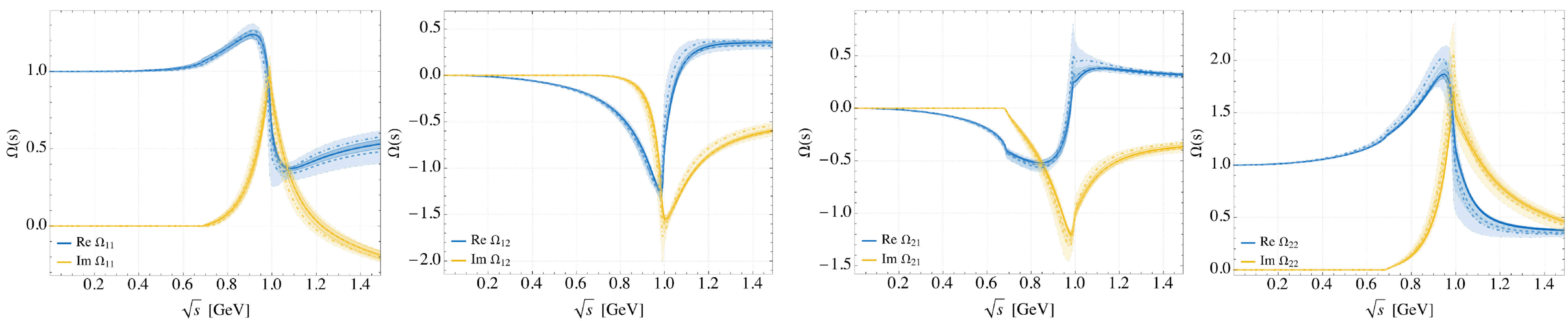
Subtracted dispersion relation for $\gamma\gamma \to \pi\pi/K\bar{K}$ (S wave) [Garcia-Martin:2010kyn]

- Cure $(\alpha_1-\beta_1)_{\pi^0}$ by including Adler zero [Dai:2016ytz]
- ! $d\sigma/d\cos\theta$ from BESIII is crucial for pion polarizabilities [Ermolina:2024daf]

$\gamma\gamma \to \pi\eta / K\bar{K}$



- **Unsubtracted** and **Subtracted** dispersion relations for $\gamma\gamma \to \pi\eta / K\bar{K}$ yield very similar hadronic $\pi\eta / K\bar{K}$ amplitudes

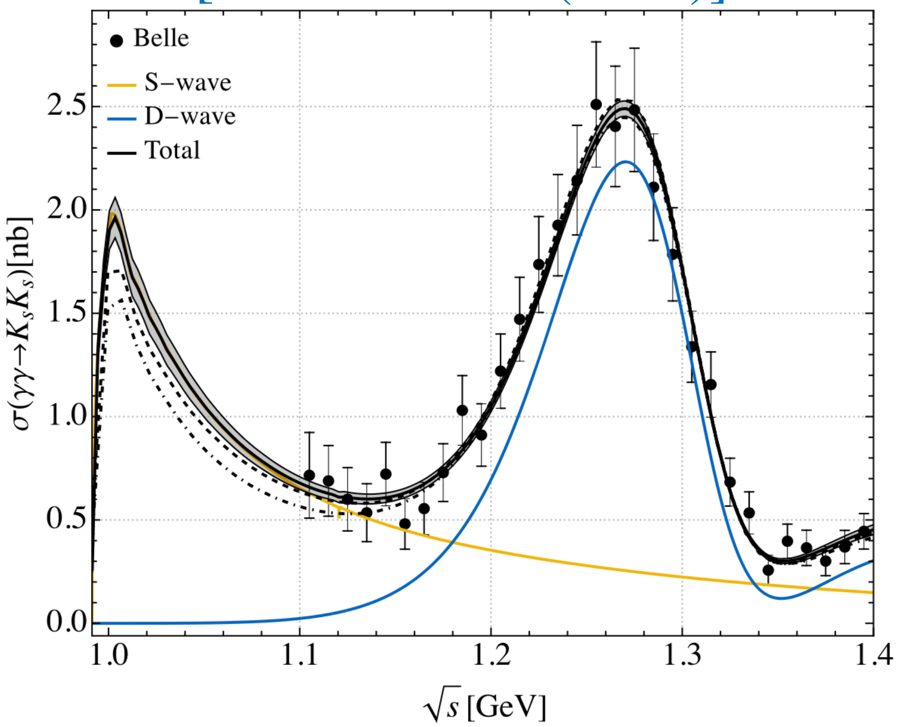


$a_0(980)$		Pole Position (MeV)	$\pi\eta$ (GeV)	$K\bar{K}$ (GeV)	$\gamma\gamma$ (MeV)
Deineka et al. 2024	RSII	$1047(18) - i\,72(17)$	3.8(3)	5.2(4)	7.3(5)
	RSIII	$930(25) - i\,80(10)$	2.9(1)	2.0(1)	8.9(3)
Lu et al. 2020	RSII	$1000^{+13}_{-1} - i\,37^{+13}_{-3}$	$2.2^{+0.6}_{-0.2}$	$4.0^{+0.3}_{-0.2}$	$5.0^{+0.9}_{-0.5}$

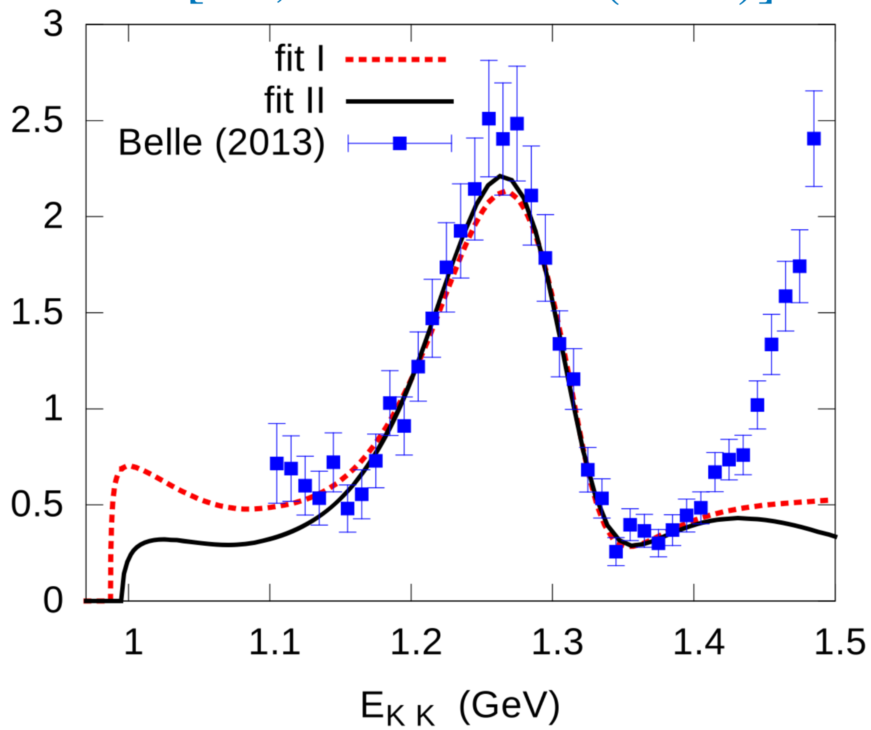
PDG (MeV):
(960...1030) – i(20...70)

Differences from other theoretical approaches

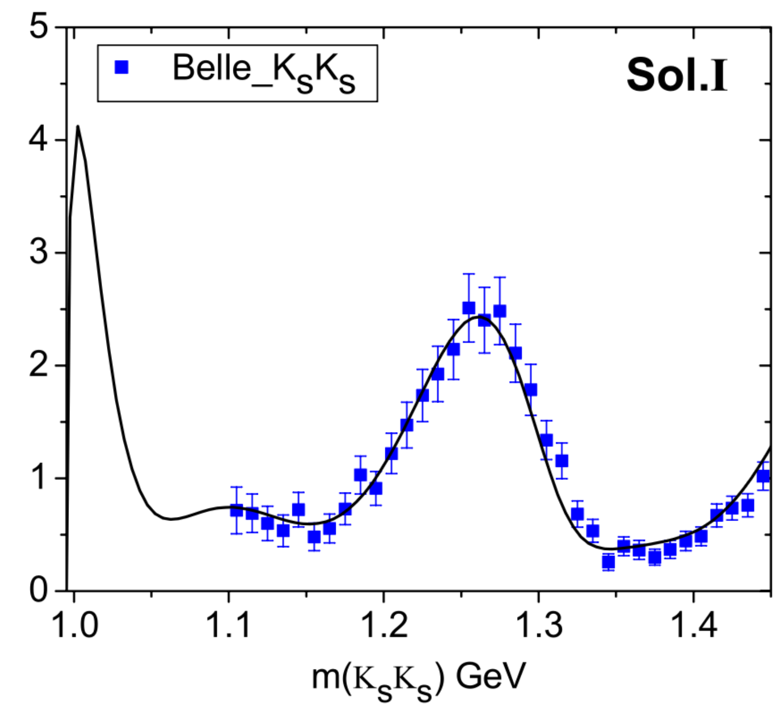
[Deineka et al. (2025)]



[Lu, Moussallam (2020)]



[Dai, Pennington (2014)]



- [Oller et al. (1998)] Summed loops using unitarized tree-level ChPT
- [Dai, Pennington (2014)] Amplitude analysis of $\gamma\gamma \to \pi\pi/K\bar{K}$
  Isovector channel parametrized phenomenologically by polynomials in s

- [Lu, Moussallam (2020)] Direct Omnès solution, **heavily** based on ChPT (only partly relies on $\gamma\gamma$ data)
  $\Omega_{ab} \sim O(1/s) \Rightarrow \delta_1 + \delta_2 \to 2\pi$, included also $a_0(1450)$ resonance
  Only subtracted DR for $\gamma\gamma \to \pi^0\eta/K\bar{K}$ is possible
  No prediction for $\gamma^*\gamma^* \to \pi^0\eta/K\bar{K}$

$$\Omega_{ab}(s) = \sum_c \int_{s_{th}}^{\infty} \frac{ds'}{\pi} \frac{t^*_{ac}(s')\,\rho_c(s')\,\Omega_{cb}(s')}{s'-s}$$

- [Deineka et al. (2025)] Dispersive (N/D) method to obtain Omnès matrix, $\Omega_{ab} \sim O(s^0)$, minimum constraints from ChPT: Adler zero $\pi\eta \to K\bar{K}$, $t_{\pi\eta\to\pi\eta}(s_{th})$, $t_{\pi\eta\to K\bar{K}}(s_{th})$
  Unsubtracted DR for $\gamma\gamma \to \pi^0\eta/K\bar{K}$ is possible
  Prediction for $\gamma^*\gamma^* \to \pi^0\eta/K\bar{K}$

$$\Omega_{ab}(s) = \delta_{ab} + \frac{s}{\pi} \sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'} \frac{t^*_{ac}(s')\,\rho_c(s')\,\Omega_{cb}(s')}{s'-s}$$

- The polarized cross section for the process $e^+e^- \to e^+e^- X$ is given by

$$d\sigma_{h_1 h_2} = \frac{1}{F}\, d\mathrm{Lips} \boxed{\sum_{h_1', h_2'} |\mathcal{M}|^2}$$

$\to L_{h_1,\mu\mu'} L_{h_2,\nu\nu'} H^{\mu\nu} (H^{\mu'\nu'})^*$ — Lorentz-covariant (**Ekhara 3.2**)

$\to \sum_{\lambda_1,\lambda_2,\lambda_1',\lambda_2'} \rho_{h_1}^{\lambda_1\lambda_1'} \rho_{h_2}^{\lambda_2\lambda_2'} H_{\lambda_1\lambda_2} H^*_{\lambda_1'\lambda_2'}$ — equivalent form (**HadroTOPS**)

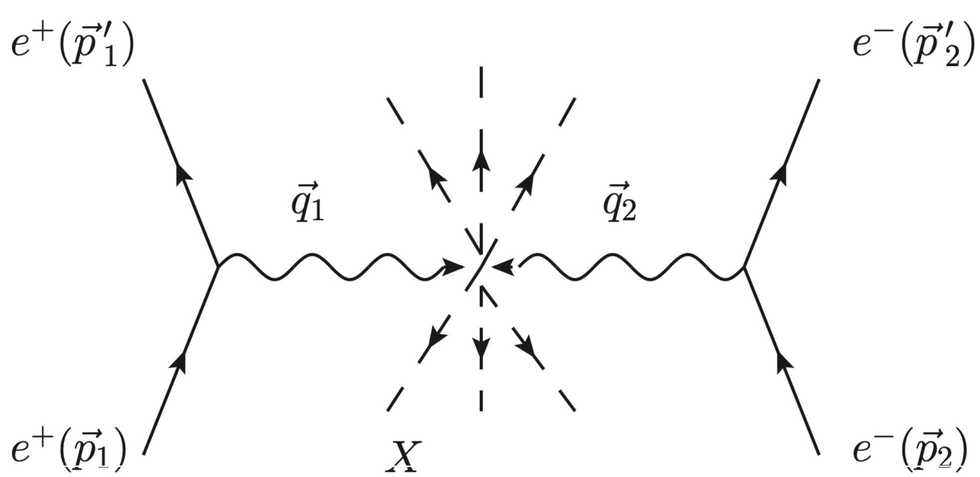
- The central object in the HadroTOPS framework is the imaginary part of the forward LbL amplitude $\mathrm{Im}\, M_{\lambda_1'\lambda_2',\lambda_1\lambda_2}$, which, according to unitarity is given by

$$\mathrm{Im}\, M_{\lambda_1'\lambda_2',\lambda_1\lambda_2} = \frac{1}{2} \sum_X \int d\Gamma_X (2\pi)^4 \delta^4(q_1 + q_2 - p_X) H_{\lambda_1\lambda_2} H^*_{\lambda_1'\lambda_2'}$$

$$= \sum_X 2\sqrt{X_{\text{flux}}}\, \sigma_{\lambda_1'\lambda_2',\lambda_1\lambda_2}(\gamma\gamma \to X) = \sum_{X=\pi,\ldots} + \sum_{X=\pi\pi,\ldots} + \ldots$$

- The **inclusive** $e^+e^- \to e^+e^- X$ cross-section can be expressed compactly in terms of 8 independent response functions $\mathrm{Im}\, M_{++,++}, \ldots, \mathrm{Im}\, M_{++,00}$ [Bonneau:1973kg, Budnev:1974de]

$$d\sigma_{h_1 h_2} = \frac{\alpha^2}{8\pi^4 Q_1^2 Q_2^2} \frac{\sqrt{X}}{s(1-4m_e^2/s)^{1/2}} \frac{d^3\vec{p}_1{}'}{E_1'} \frac{d^3\vec{p}_2{}'}{E_2'} \Big\{ 4\,\rho_1^{++}\,\rho_2^{++}\,\frac{1}{2}\left(\sigma_0+\sigma_2\right) + \rho_1^{00}\,\rho_2^{00}\,\sigma_{LL}$$
$$\ldots$$
$$+(\ldots)\cos\tilde{\phi}\,\frac{1}{2}\left(\tau_0+\tau_1\right) + h_1\,h_2\,(\ldots)\Big\}$$



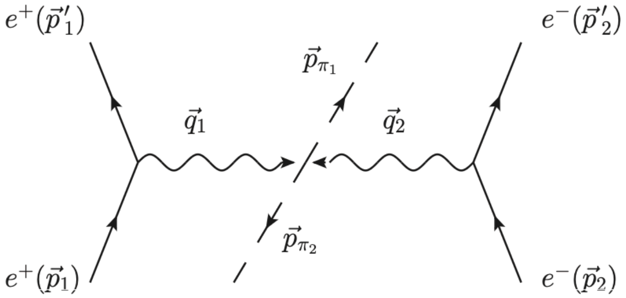
inclusive (sum over X)

$$\sigma_0 = \frac{1}{2\sqrt{X}}\,\mathrm{Im}\,M_{++,++} = \sum_X \ldots$$
$$\ldots$$
$$\tau_0 = \frac{1}{2\sqrt{X}}\,\mathrm{Im}\,M_{++,00} = \sum_X \ldots$$

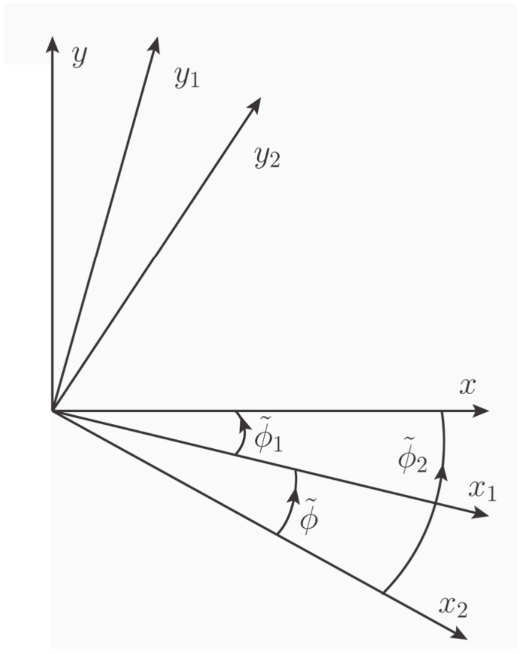
for $X = \pi\pi$
it is complex

Theory contribution to HadroTOPS

- For the **exclusive** process $e^+e^- \to e^+e^-\pi\pi$ the expressed becomes much longer ($15^{(\text{unpol})} + 10 = 25$ differential hadronic response functions) with additional differential dependence on azimuthal angles [Lellmann et al. (2025)]



$$
\begin{aligned}
d\sigma^{(unpol)} =\ & \frac{\alpha^2}{8\pi^4\, Q_1^2\, Q_2^2} \frac{\sqrt{X}}{s(1-4m^2/s)^{1/2}} \frac{d^3\vec{p}_1^{\,\prime}}{E_1'} \frac{d^3\vec{p}_2^{\,\prime}}{E_2'} \frac{d\Omega_\pi}{2\pi} \frac{4}{(1-\varepsilon_1)(1-\varepsilon_2)} \\
& \times \left\{ \frac{1}{2}\left(\frac{d\sigma_0}{d\cos\theta_\pi} + \frac{d\sigma_2}{d\cos\theta_\pi} \right) + \left[\varepsilon_1 + \frac{2m^2}{Q_1^2}(1-\varepsilon_1) \right] \left[\varepsilon_2 + \frac{2m^2}{Q_2^2}(1-\varepsilon_2) \right] \frac{d\sigma_{LL}}{d\cos\theta_\pi} \right. \\
& + \left[\varepsilon_2 + \frac{2m^2}{Q_2^2}(1-\varepsilon_2) \right] \left(1 + \varepsilon_1 \cos\left(2\tilde{\phi}_1\right) \right) \frac{d\sigma_{TL}}{d\cos\theta_\pi} + \left[\varepsilon_1 + \frac{2m^2}{Q_1^2}(1-\varepsilon_1) \right] \left(1 + \varepsilon_2 \cos\left(2\tilde{\phi}_2\right) \right) \frac{d\sigma_{LT}}{d\cos\theta_\pi} \\
& + \frac{1}{2}\varepsilon_1\varepsilon_2 \left[\cos 2(\tilde{\phi}_2 - \tilde{\phi}_1) \frac{d\sigma_0}{d\cos\theta_\pi} + \cos 2(\tilde{\phi}_1 + \tilde{\phi}_2) \frac{d\sigma_2}{d\cos\theta_\pi} \right] - \left[\varepsilon_1 \cos\left(2\tilde{\phi}_1\right) + \varepsilon_2 \cos\left(2\tilde{\phi}_2\right) \right] \frac{d\tau_{T2}}{d\cos\theta_\pi} \\
& + \left[\varepsilon_1(1+\varepsilon_1) + \frac{4m^2}{Q_1^2}\varepsilon_1(1-\varepsilon_1) \right]^{1/2} \left[\varepsilon_2(1+\varepsilon_2) + \frac{4m^2}{Q_2^2}\varepsilon_2(1-\varepsilon_2) \right]^{1/2} \\
& \left. \times \left[\cos\left(\tilde{\phi}_2 - \tilde{\phi}_1\right) \left(\frac{d\tau_0}{d\cos\theta_\pi} + \frac{d\tau_1}{d\cos\theta_\pi} \right) + \cos\left(\tilde{\phi}_1 + \tilde{\phi}_2\right) \left(\frac{d\tau_1}{d\cos\theta_\pi} - \frac{d\tau_{L2}}{d\cos\theta_\pi} \right) \right] \cdots \right\}
\end{aligned}
$$



Planes: $e^+e^+, e^-e^-, \pi\pi$

$\tilde{\phi} = \tilde{\phi}_2 - \tilde{\phi}_1$

$d^3\vec{p}_1^{\,\prime} = |\vec{p}_1^{\,\prime}|^2\, d|\vec{p}_1^{\,\prime}|\, d\cos\theta_1\, d\tilde{\phi}_1$

$d^3\vec{p}_2^{\,\prime} = |\vec{p}_2^{\,\prime}|^2\, d|\vec{p}_2^{\,\prime}|\, d\cos\theta_2\, d\tilde{\phi}_2$

$\phi_\pi = 0$ (choice)

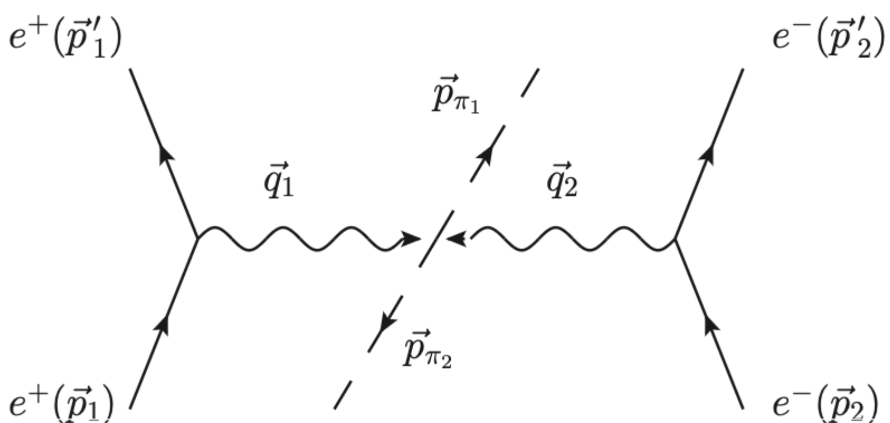
- **HadroTOPS** results are consistent with **Ekhara 3.2**

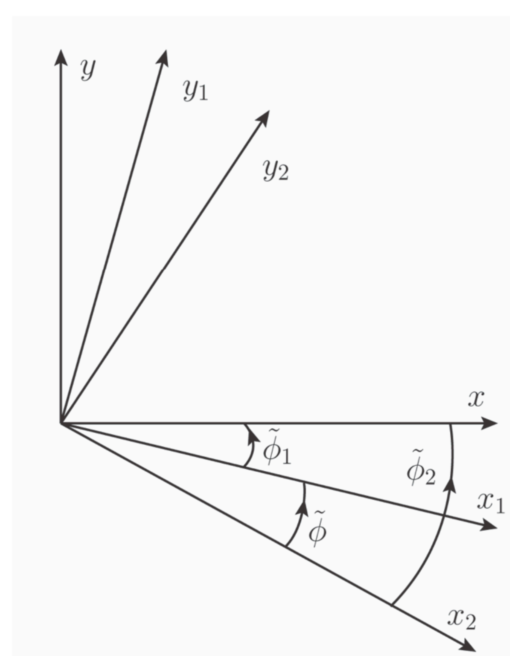
- Integrating over $\tilde{\phi}_2$ and taking the limit $Q_2 \to 0$

$$d\sigma^{(unpol)}|_{Q_2^2\to 0} = \frac{\alpha^2}{8\pi^4\, Q_1^2\, Q_2^2}\,\frac{\sqrt{X}}{s(1-4m^2/s)^{1/2}}\,\frac{d^3\vec{p}_1{}'}{E_1'}\,\frac{d^3\vec{p}_2{}'}{E_2'}\,\frac{d\Omega_\pi}{2\pi}\,\frac{4}{(1-\varepsilon_1)(1-\varepsilon_2)}$$

$$\left\{\frac{1}{2}\left(\frac{d\sigma_0}{d\cos\theta_\pi}+\frac{d\sigma_2}{d\cos\theta_\pi}\right)+\ldots-(\ldots)\cos(2\tilde{\phi}_1)\frac{d\tau_{T2}}{d\cos\theta_\pi}+(\ldots)\cos\tilde{\phi}_1\left(\frac{d\tau_{-12}}{d\cos\theta_\pi}-\frac{d\tau_{-1T}}{d\cos\theta_\pi}\right)\right\}$$

- **Novel way** to experimentally **extract additional information** about response functions of $\gamma^*\gamma^* \to \pi\pi$. For example





$$\frac{d\sigma_0}{d\cos\theta_\pi} \propto \left|H_{++}\right|^2$$

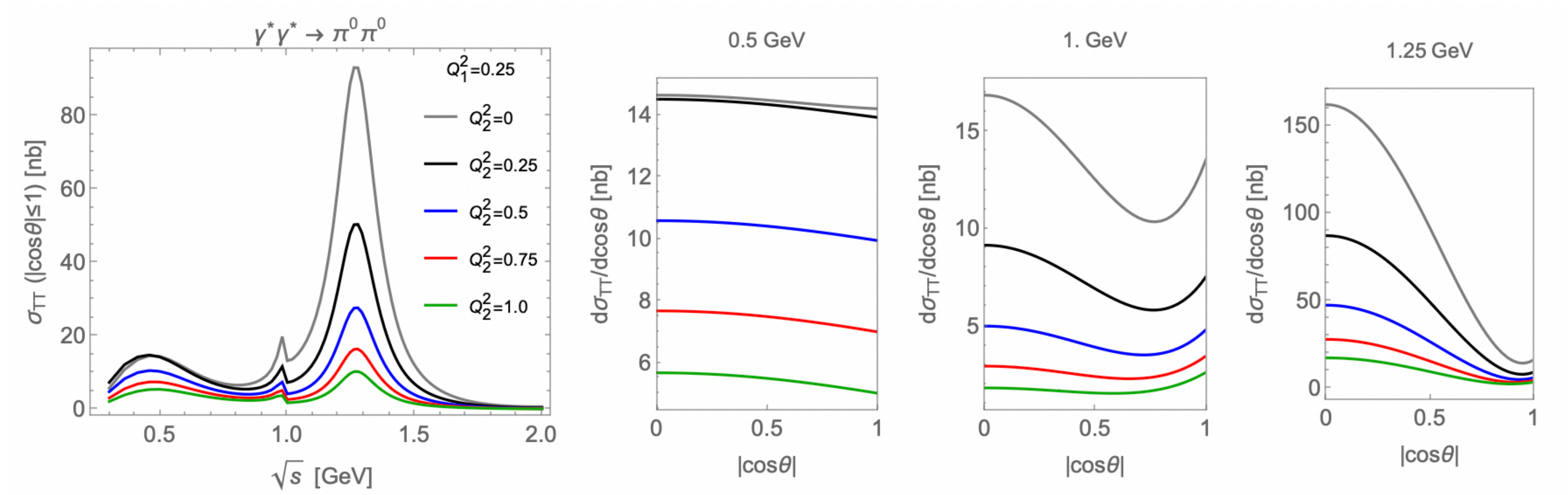
$$\frac{d\sigma_2}{d\cos\theta_\pi} \propto \left|H_{+-}\right|^2$$

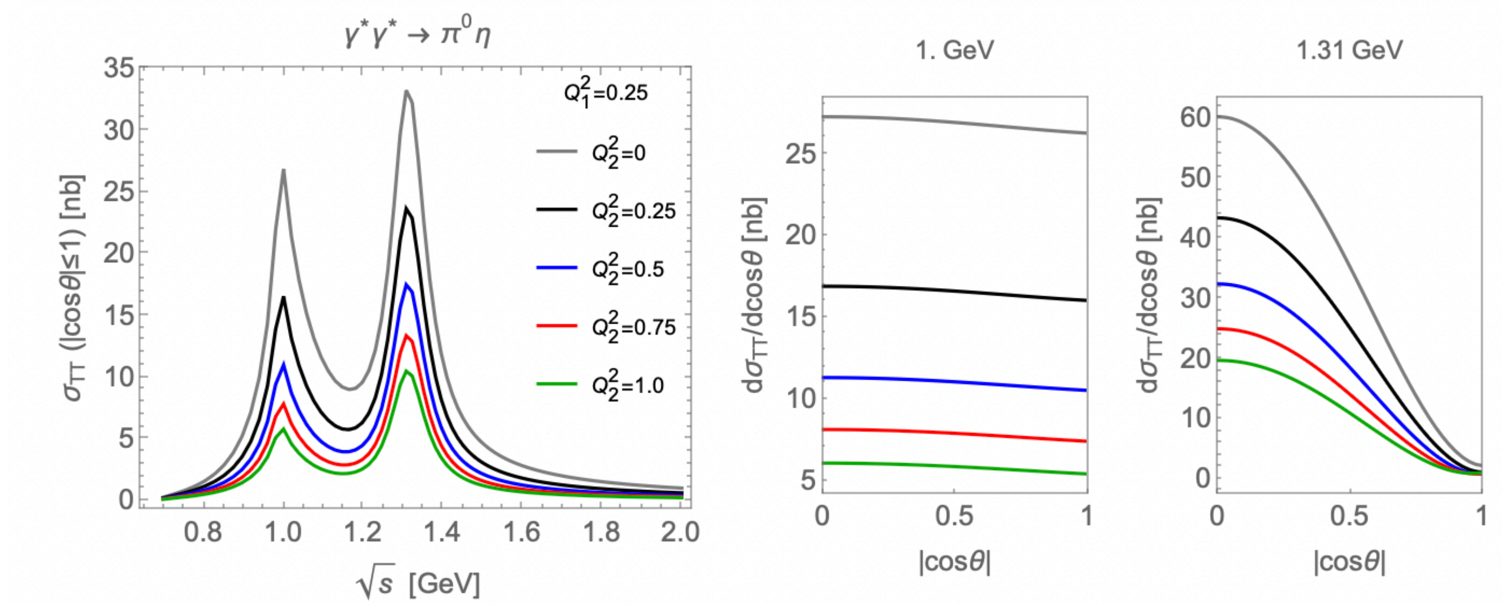
$$\frac{d\tau_{T2}}{d\cos\theta_\pi} \propto \mathrm{Re}\left(H_{++}^*\, H_{+-}\right)$$

Theory contribution to HadroTOPS

- Input provided to HadroTOPS (example)

$$\frac{d\sigma_{TT}}{d\cos\theta_\pi} \equiv \frac{\beta_\pi}{32\pi\sqrt{X}}\left(\left|H_{++}\right|^2 + \left|H_{+-}\right|^2\right)$$





- For $\gamma^*\gamma^* \to \pi^0\pi^0$: $J = 0{,}2$ dispersive [Danilkin:2019opj, Danilkin:2019mhd]
- For $\gamma^*\gamma^* \to \pi^0\eta$: $J = 0$ dispersive [Deineka:2023nhu]

  $J = 2$ BreitWigner with FFs fixed from the QM [Schuler:1997yw] adopted using the $T \to \gamma^*\gamma^*$ basis [Hoferichter:2020lap]