MESON 2023

17th International Workshop on Meson Physics

22nd - 27th June 2023, Kraków, Poland

Hadronic contribution to the muon g−2 with emphasis on photon−photon fusion processes

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OUTLINE

- General introduction of muon (g-2)
- Hadronic contributions to a_μ: HVP and HLbL
- Photon-photon fusion studies
- Summary and outlook

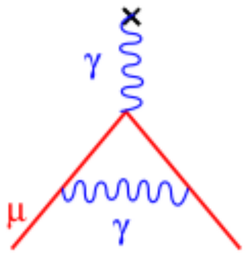
The Muon (g-2)

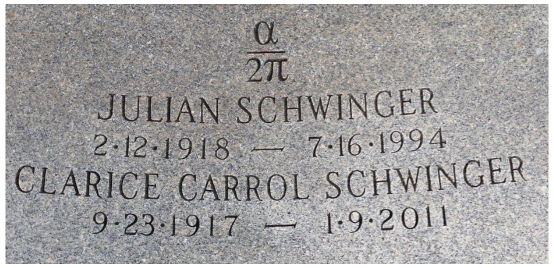
Magnetic moment of lepton

- Gyromagnetic factor:

$$\boldsymbol{\mu}_\ell = g_\ell \frac{e_\ell}{2m_\ell} \boldsymbol{S}, \quad \boldsymbol{S} = \frac{\boldsymbol{\sigma}}{2}\hbar$$

✓ $g_\ell = 2\,_{(\textbf{Dirac})} + 2\,a_\ell\,_{(\textbf{quantum correction})}$

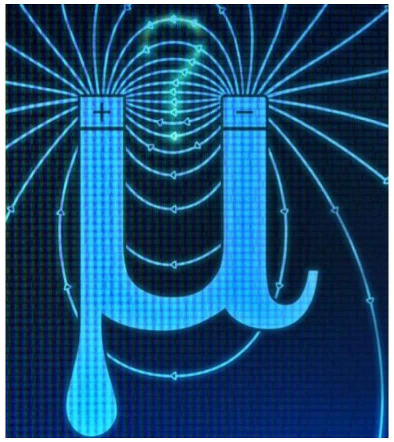




- **Anomalous magnetic moment:**

$$\boxed{a_\ell = \frac{1}{2}(g_\ell - 2)}$$

Why muon (g-2): a_μ?



Dani Zemba

- Can be measured with unprecedented precision
- Can be calculated very precisely in Standard Model

✓ **Excellent test of SM**

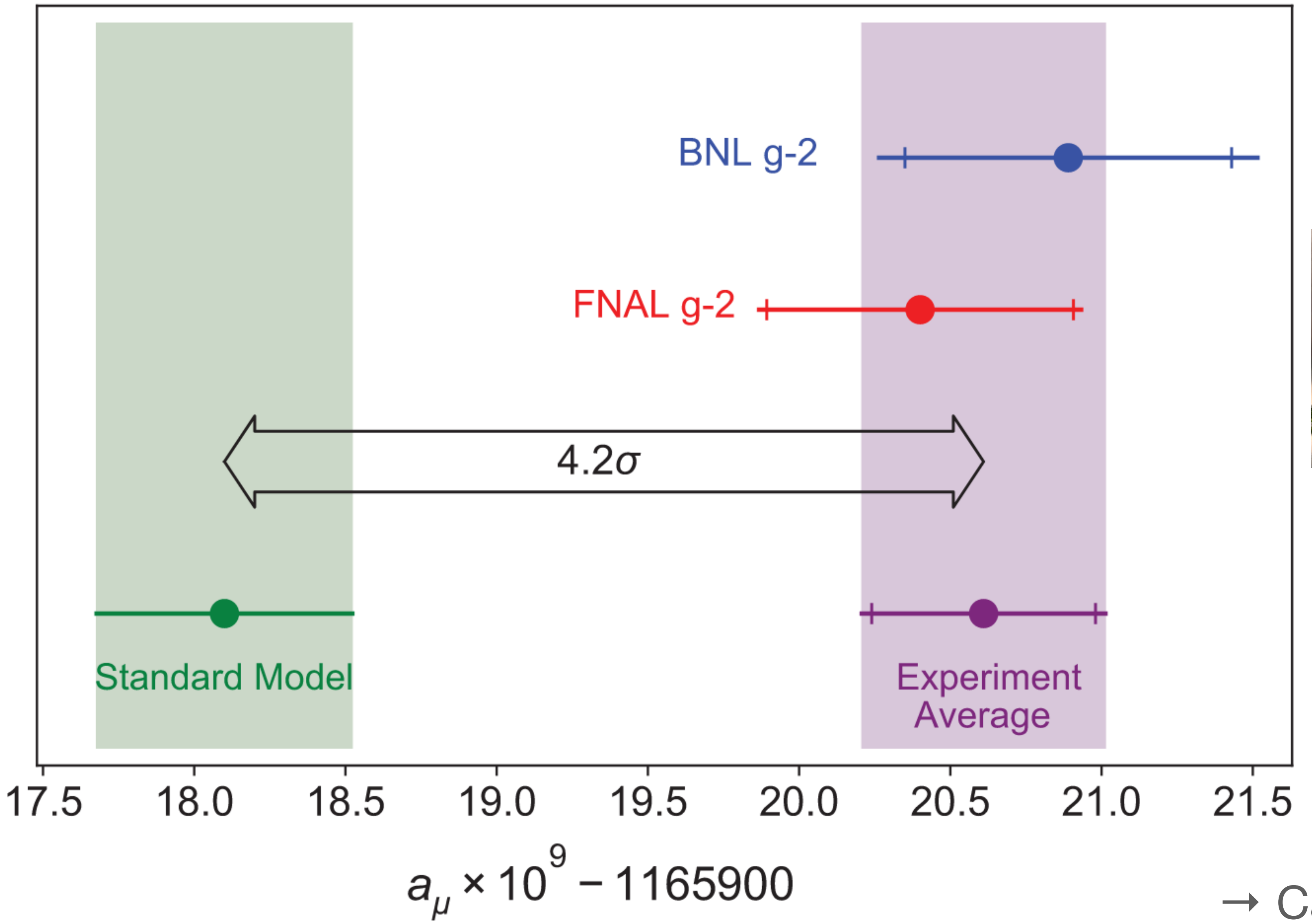
- Highly sensitive to new physics

✓ a_μ is $(m_\mu/m_e)^2 \sim 40{,}000$ times more sensitive than a_e

Muon (g−2): experiment vs. SM prediction

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (25.1 \pm 5.9) \cdot 10^{-10}$$

White Paper: *Phys. Rept. 887, 1 (2020)*



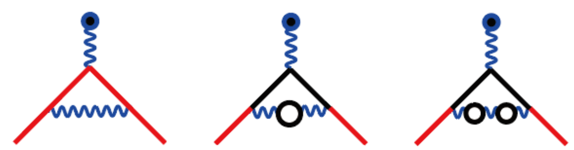
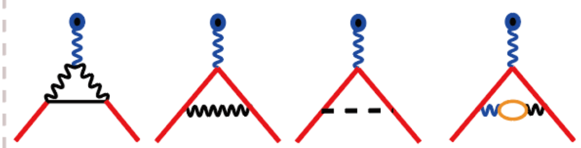
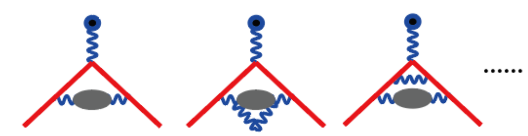
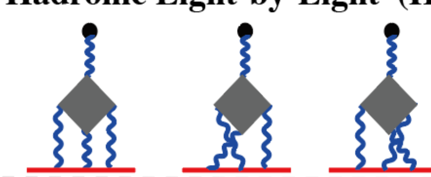
→ Call for New Physics?

- **FNAL precision goal:** $\Delta a_\mu^{\text{exp}} \approx \pm 1.6 \cdot 10^{-10}$ (1/4 of current error)
- **Efforts are required to decrease the theoretical error of the SM value**

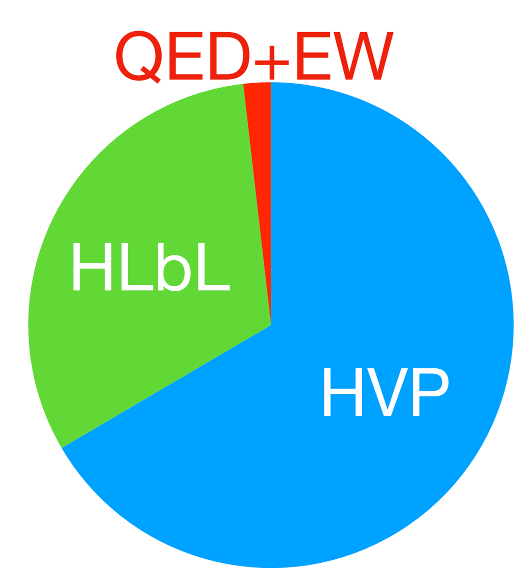
Muon (g−2): Standard Model prediction

$$a_\mu^{\mathrm{SM}} = a_\mu^{\mathrm{QED}} + a_\mu^{\mathrm{EW}} + a_\mu^{\mathrm{Had}}$$

WP(2020) $\times 10^{-11}$

QED *fully perturbative*	$\left(\frac{\alpha}{\pi}\right)^1 + \left(\frac{\alpha}{\pi}\right)^2 + \left(\frac{\alpha}{\pi}\right)^3 + \left(\frac{\alpha}{\pi}\right)^4 + \left(\frac{\alpha}{\pi}\right)^5$		116584718.931 ± 0.104
EW *perturbative + (small) non-pert.*	1-loop + 2-loop		153.6 ± 1.0
Had *fully non-perturbative*	$\left(\frac{\alpha}{\pi}\right)^2 + \left(\frac{\alpha}{\pi}\right)^3 + \left(\frac{\alpha}{\pi}\right)^4$	**Hadronic vacuum polarization (HVP)** 	6845 ± 40
	Hadronic Light-by-Light (HLbL) $\left(\frac{\alpha}{\pi}\right)^3$	**Hadronic Light-by-Light (HLbL)** 	92 ± 18

- QED provides more than 99.99% of the total value
  - with very small error (zero confront with expt. uncertainty)
- Errors are dominated by hadronic contribution
  - HLbL has larger relative uncertainty than HVP

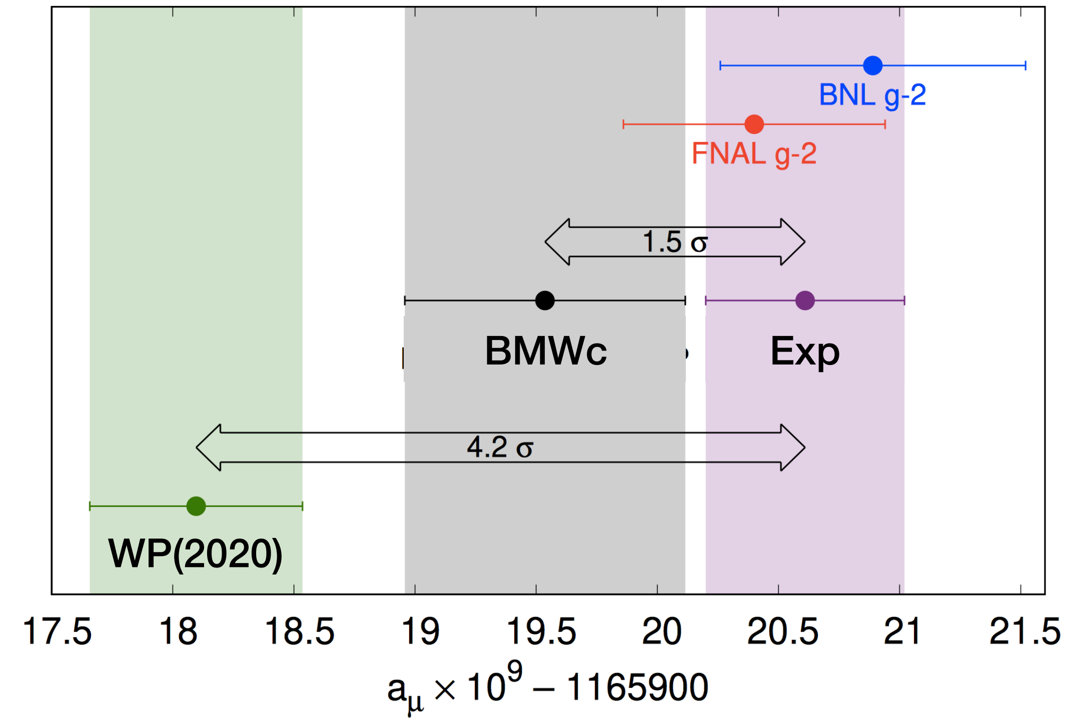


Hadronic contribution to a_μ

HVP tensions

- Theory(data-driven) vs. Lattice QCD
  - ✓ WP $a_\mu^{\mathrm{LO-HVP}} = (693.1 \pm 4.0) \cdot 10^{-10}$
  - ✓ BMWc $a_\mu^{\mathrm{LO-HVP}} = (707.5 \pm 5.5) \cdot 10^{-10}$

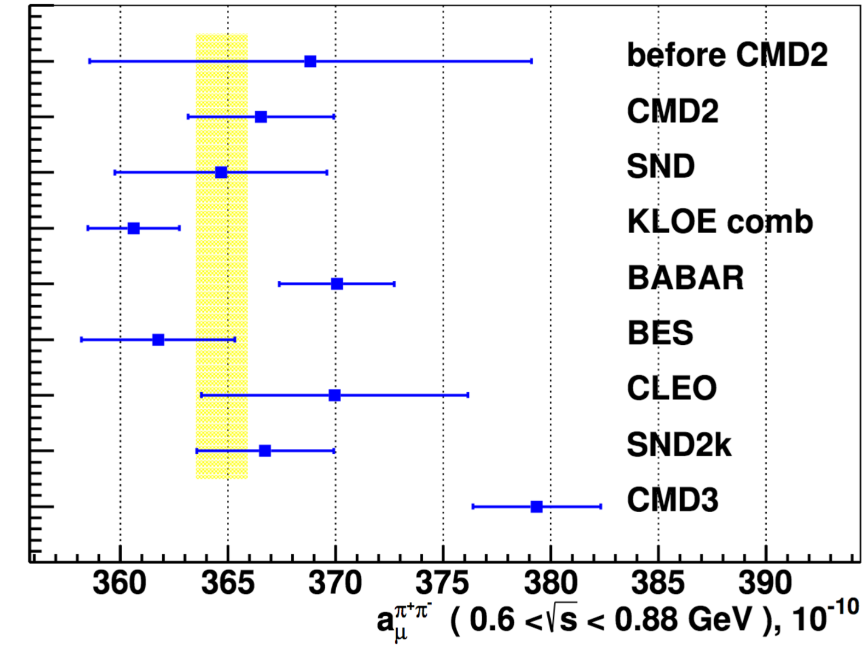
BMWc, Nature 593, 7857(2021)



$$a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{SM}}\Big|^{\mathrm{LO,HVP}}_{\mathrm{BMWc}} = (10.7 \pm 7.0) \cdot 10^{-10} \quad [1.5\sigma]$$

No new physics

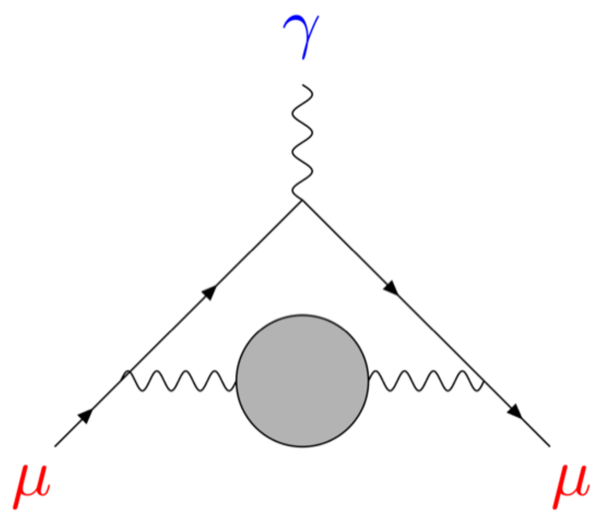
- Experiments ($e^+e^- \to \pi^+\pi^-$) tension
  - ✓ CMD3 vs CMD2 and others *arXiv:2302.08834*



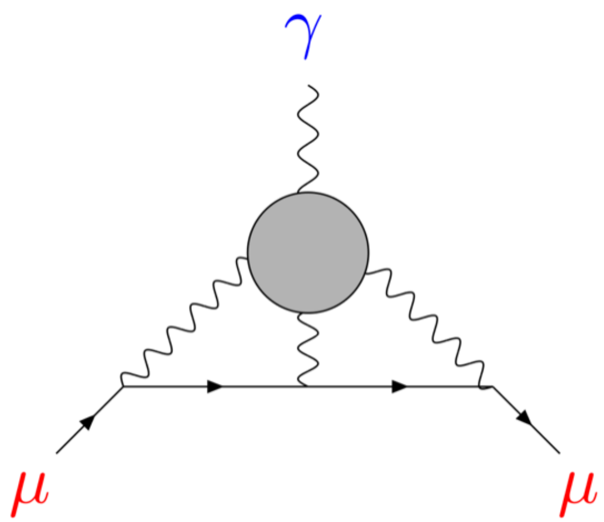
HLbL (four-point function) is more complicated than HVP

Hadronic contribution needs further studies!

Hadronic contributions



Hadronic vacuum polarisation

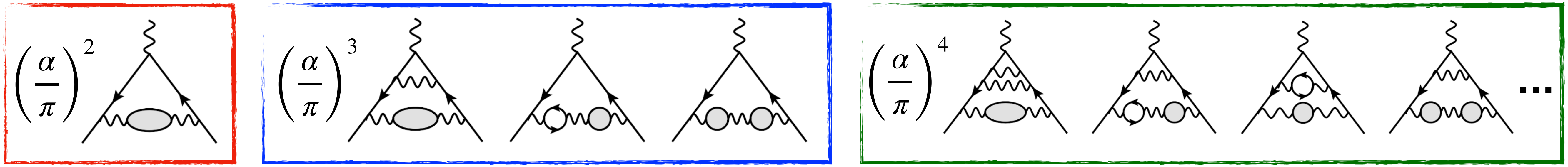


Hadronic light-by-light scattering

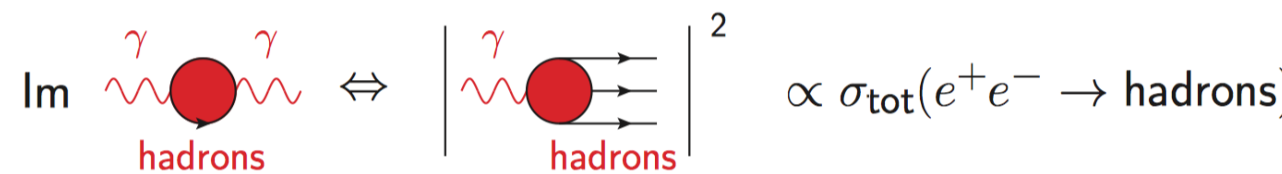
- Non-perturbative regime of QCD

- Mitigate the uncertainties -> data-driven dispersive approach

Hadronic vacuum polarization



- **Data-driven approach**
  - Based on analyticity and unitarity

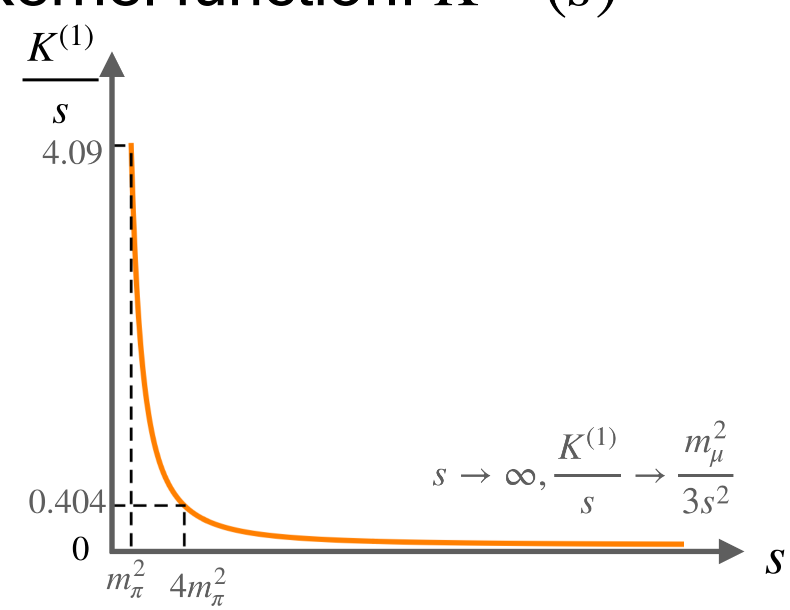


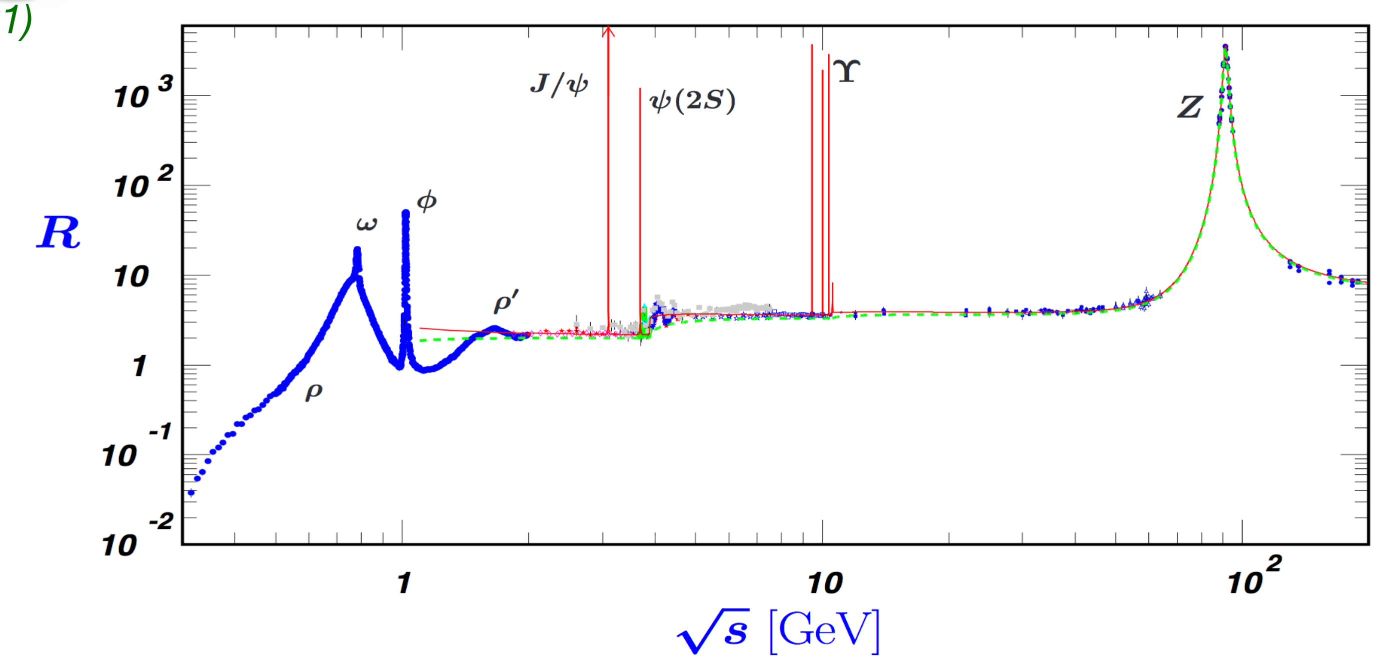
  - **Dispersion integrals over cross section of e^+e^- annihilation**

$$a_\mu^{\text{LO-HVP}} = \frac{\alpha^2}{3\pi^2}\int_{M_\pi^2}^{\infty} ds\, \frac{K^{(1)}(s)}{s} R(s) \qquad R(s) = \frac{\sigma^0\left(e^+e^- \to \text{hadrons}(+\gamma)\right)}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

J. Phys. Radium 22, 121 (1961)

✓ Kernel function: $K^{(1)}(s)$

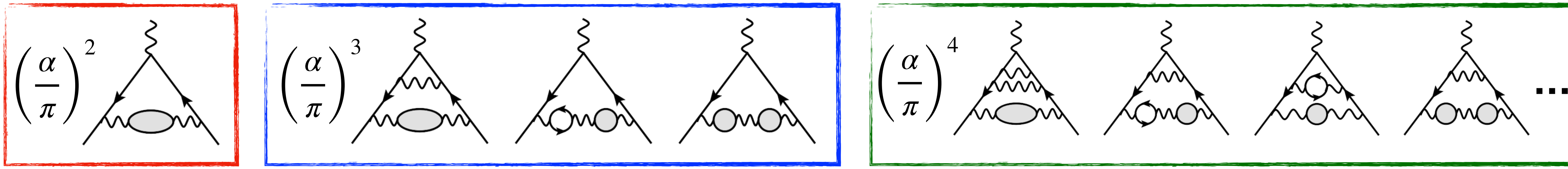




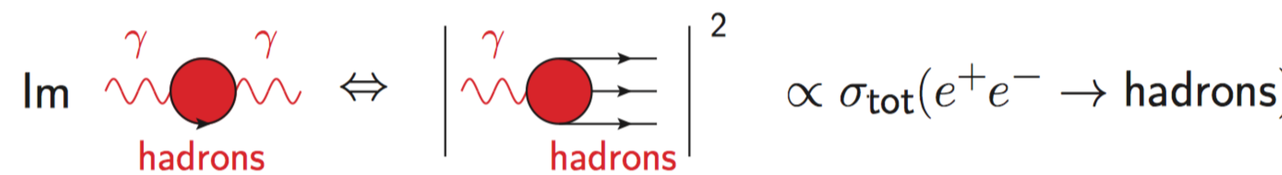
PDG(2022)

✓ **Main contribution from low energies:** $\sim 75\ \%$ from $\pi^+\pi^-$ $[\rho(770)]$ channel

Hadronic vacuum polarization



- Data-driven approach
  - Based on analyticity and unitarity



  - **Dispersion integrals over cross section of e^+e^- annihilation**

White Paper: T. Aoyama et al., Phys. Rept. 887, 1 (2020)

LO-HVP	$6931(40) \cdot 10^{-11}$
NLO-HVP	$-98.3(7) \cdot 10^{-11}$
NNLO-HVP	$12.4(1) \cdot 10^{-11}$

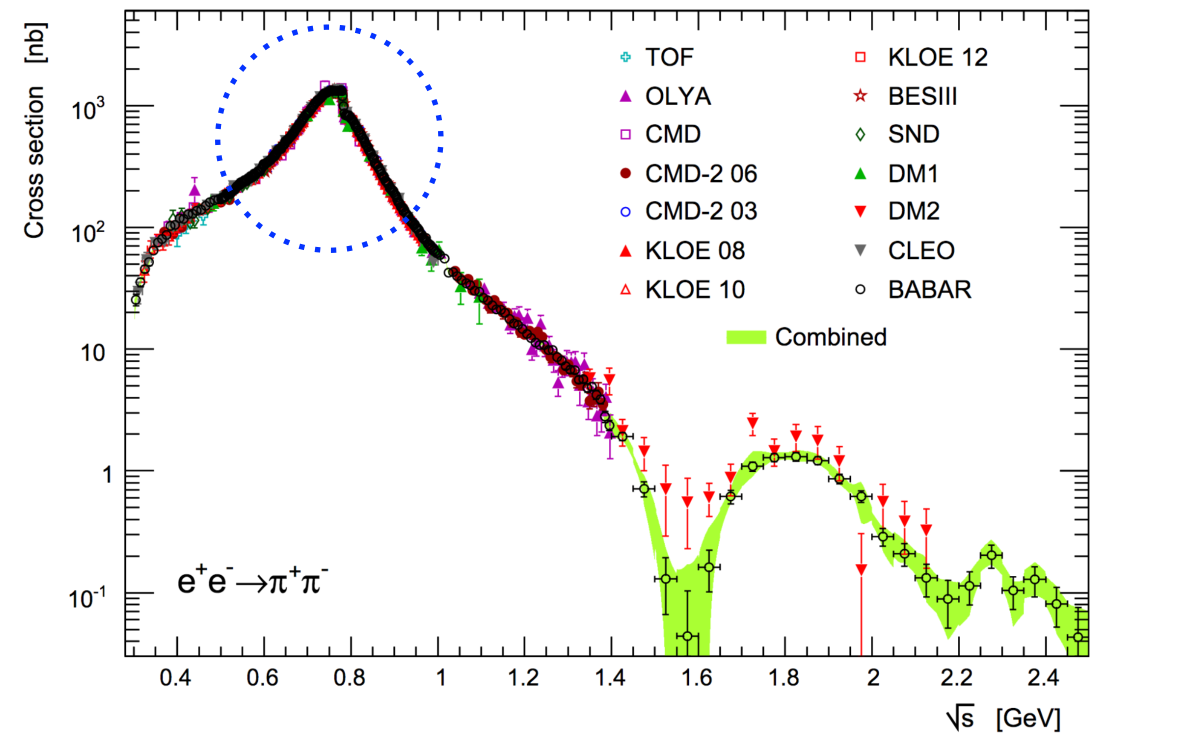
Uncertainty dominated by the total cross section of $e^+e^- \to \pi^+\pi^-$ channel

$$a_\mu^{\pi^+\pi^-} = 5060(34) \cdot 10^{-11}$$

LO-HVP: pion-pion channel

- Tension for $\sigma(e^+e^- \to \pi^+\pi^-)$ data
  - BaBar and KLOE
    - ✓ Most sub-percent precision
    - ✓ $2.9\,\sigma$ discrepancy on $a_\mu^{\pi^+\pi^-}$
  - CMD-3 *arXiv:2302.08834*
    - ✓ **Further tension**
    - ✓ CMD-3 and others are not consistent!

ρ, ω peak $\sim 5\,\%$ uncertainty



- Challenges of dispersive LO-HVP
  - Understand the above tensions
  - Understand the correlations when combining data from different expts.
  - $\pi^+\pi^-$ production cross section $\sim 0.2\,\%$ overall systematical error
    - ✓ Conform to the precision goal of FNAL-E989: $\Delta a_\mu^{\rm exp} \approx \pm 1.6 \cdot 10^{-10}$

LO-HVP: data-driven vs LQCD

Lattice QCD

- First principle to evaluate HVP

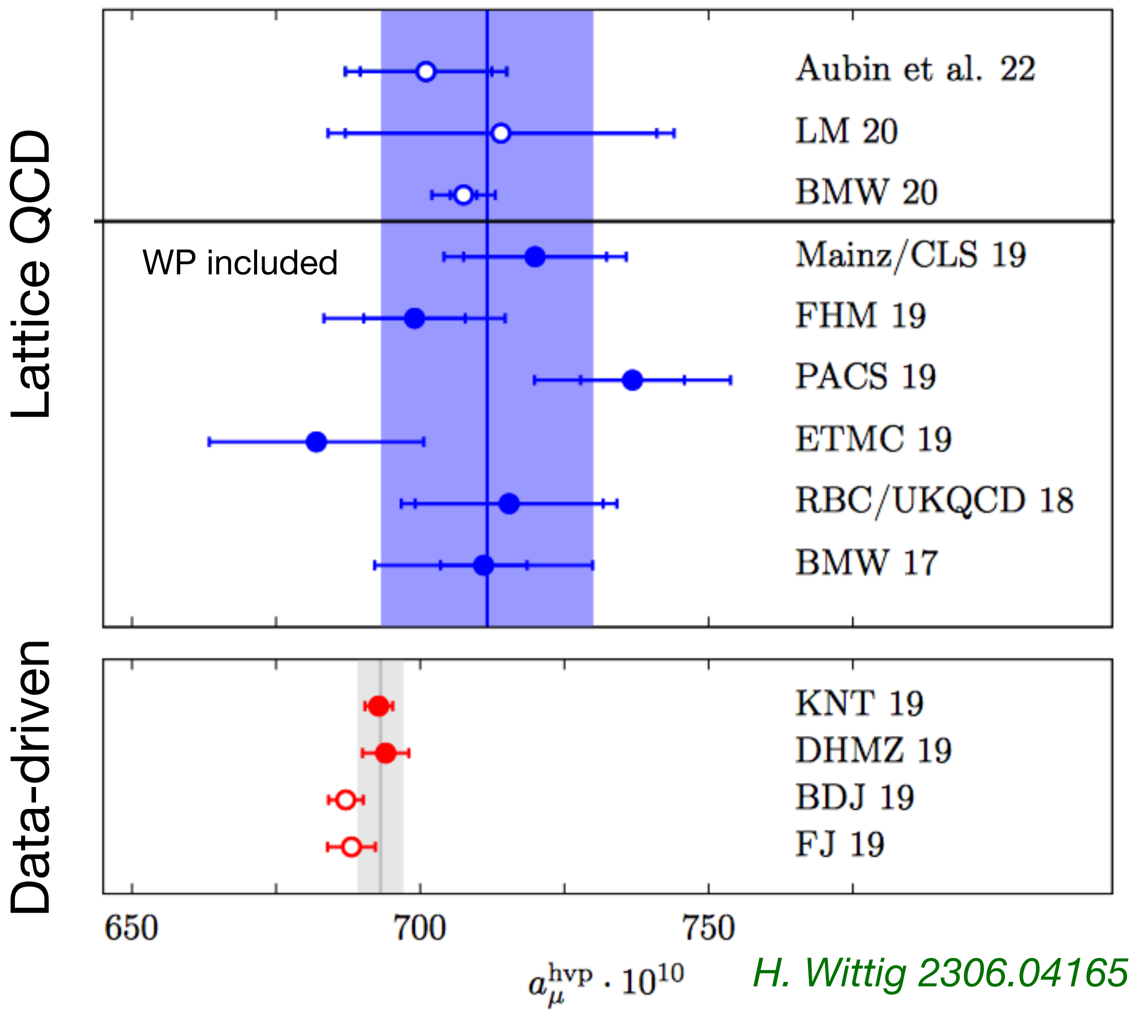
$$a_\mu^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt \tilde{K}(t) G(t)$$

✓ Correlator of the electromagnetic current

$$G(t) = -a^3 \sum_{\boldsymbol{x}} \langle J_k^{\text{e.m.}}(\boldsymbol{x}, t) J_k^{\text{e.m.}}(0) \rangle$$

✓ $\tilde{K}(t)$: known kernel function

- No reliance on experimental data



H. Wittig 2306.04165

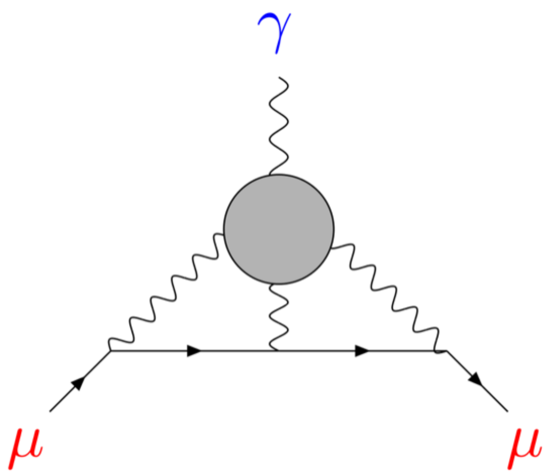
However, data-driven vs BMWc (sub-percent precision)

- LO-HVP has a $2.1\,\sigma$ tension

→ Call for independent cross-check from LQCD with high precision

Hadronic light-by-light

- Start contribution at order $(\alpha/\pi)^3$



- Involves the 4th-rank tensor: $\Pi_{\mu\nu\rho\sigma}(q_1, q_2, q_3, q_4)$
  - ✓ Much more complicated than HVP
- Suppressed by an α/π factor in comparison with HVP
  - ✓ Its accuracy is $\lesssim 10\,\%$ to meet the precision goal of FNAL

- **Dispersion relation** for HLbL amplitude G. Colangelo, et al., JHEP09,091(2014); 09,074(2015)

- Construct the basis for rank 4 polarization tensor (fully off-shell)
  - Lorentz invariance
  - Gauge invariance

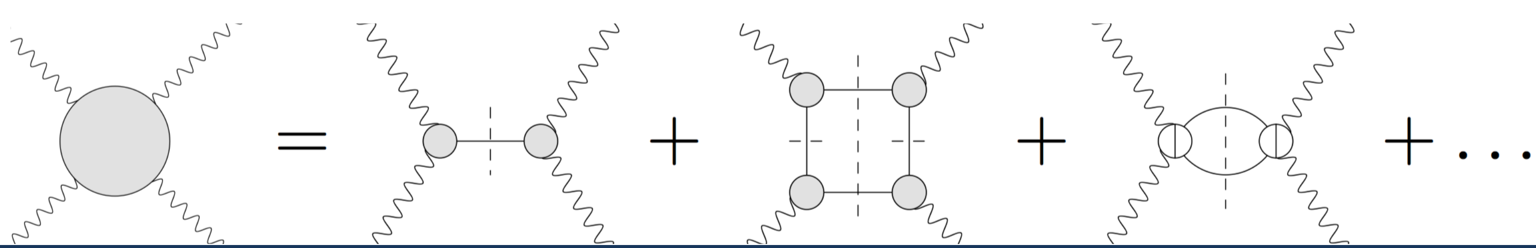
$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma}\,\Pi_i$$

- Master formula for HLbL

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \sqrt{1-\tau^2}\,Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i\,(Q_1, Q_2, \tau)\,\bar{\Pi}_i\,(Q_1, Q_2, \tau)$$

✓ T_i Integral kernels, known

✓ $\bar{\Pi}_i$ Parameterize the hadronic states in HLbL



Hadronic light-by-light

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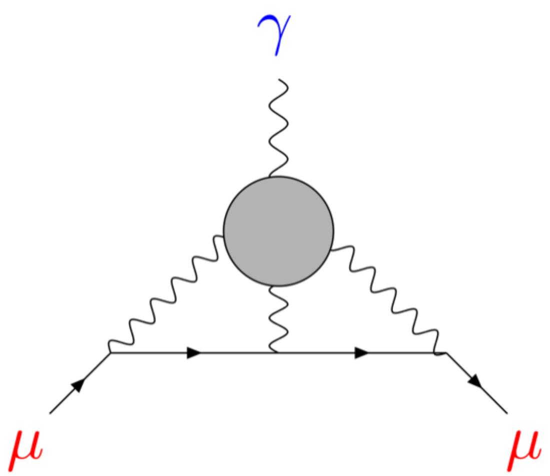

- Involves the 4th-rank tensor: $\Pi_{\mu\nu\rho\sigma}(q_1, q_2, q_3, q_4)$
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  - ✓ Its accuracy is $\leq 10\,\%$ to meet the precision goal of FNAL

- **Dispersion relation** for HLbL amplitude *G. Colangelo, et al., JHEP09,091(2014); 09,074(2015)*
  - Construct the basis for rank 4 polarization tensor (fully off-shell)
    - Lorentz invariance
    - Gauge invariance

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma}\,\Pi_i$$

New insight on tensor with triangle kinematics

J. Lüdtke, et al., 2302.12264

  - Master formula for HLbL

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \sqrt{1-\tau^2}\,Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i\left(Q_1, Q_2, \tau\right)\bar{\Pi}_i\left(Q_1, Q_2, \tau\right)$$

✓ T_i Integral kernels, known

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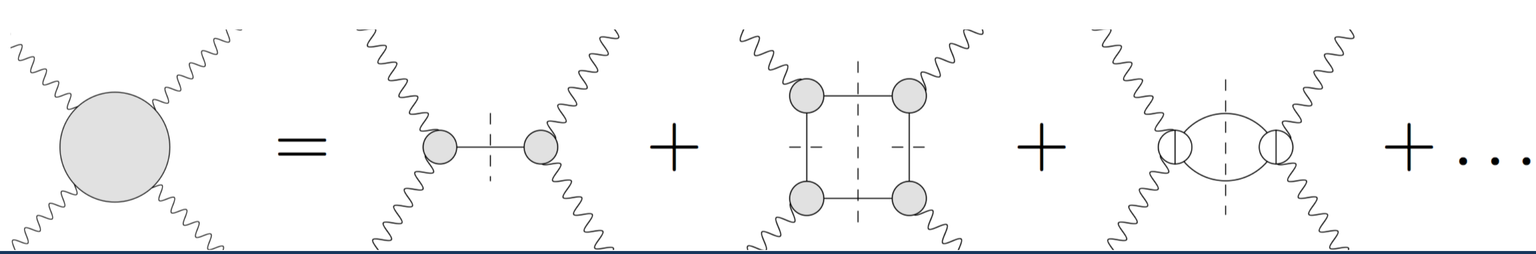
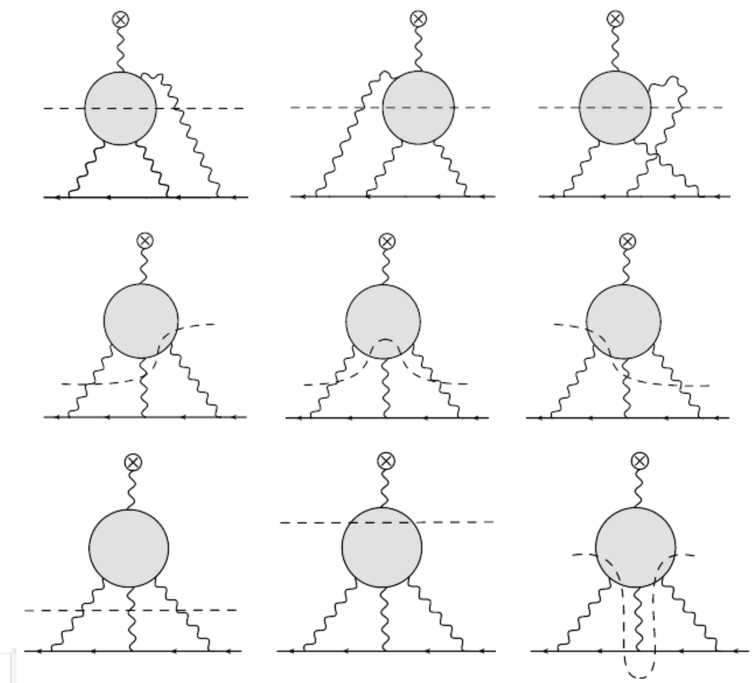


Dispersion relation for HLbL contribution

- Indirect method: DR for Pauli form factor F_2 V. Pauk, M. Vanderhaeghen, PRD90,113012 (2014)

$$a_\mu^{\mathrm{HLbL}} = F_2(0) = \frac{1}{2\pi i}\int_0^\infty \frac{dk^2}{k^2}\,\mathrm{Disc}_{k^2} F_2\left(k^2\right)$$

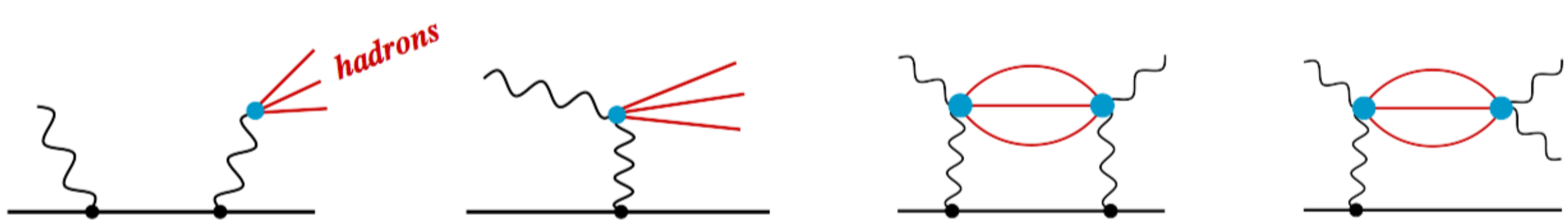


  - $F_2(k^2)$ is obtained from the two-loop integral
  - The discontinuity of the form factor $\mathrm{Disc}_{k^2} F_2(k^2)$

- Schwinger sum rule F. Hagelstein, V. Pascalutsa, PRL120,072002(2018)

$$a_\mu = \frac{m_\mu^2}{\pi^2\alpha}\int_{\nu_0}^\infty d\nu \left[\frac{\sigma_{LT}\left(\nu, Q^2\right)}{Q}\right]_{Q^2=0}$$

  - Related to photo-absorption cross section on the muon



  - **Generalize HVP dispersive relation to HLbL**

Data-driven approach for HLbL

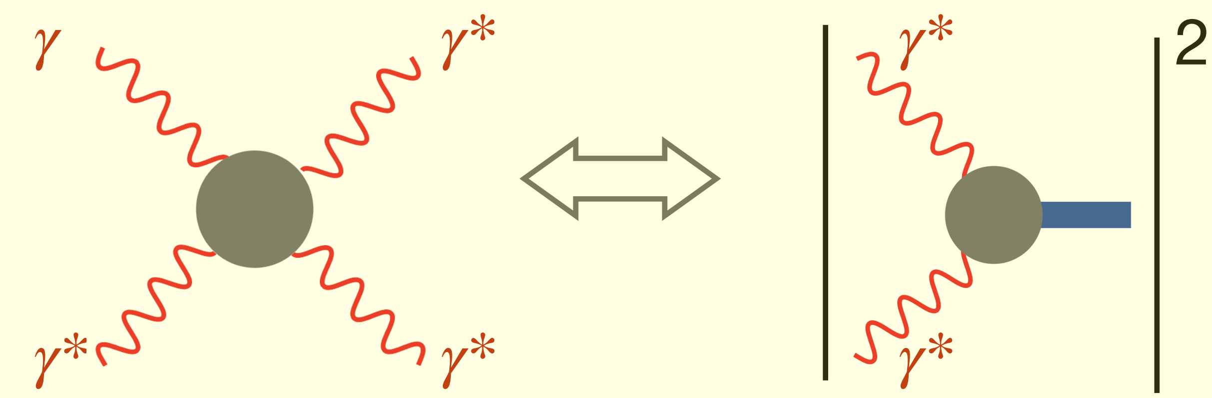
Connection between HLbL and $\gamma\gamma$-collision events

G. Colangelo, et al. (2014–2017)

- Based on the dispersion relation of HLbL

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \sqrt{1-\tau^2}\, Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i\left(Q_1, Q_2, \tau\right) \bar{\Pi}_i\left(Q_1, Q_2, \tau\right)$$

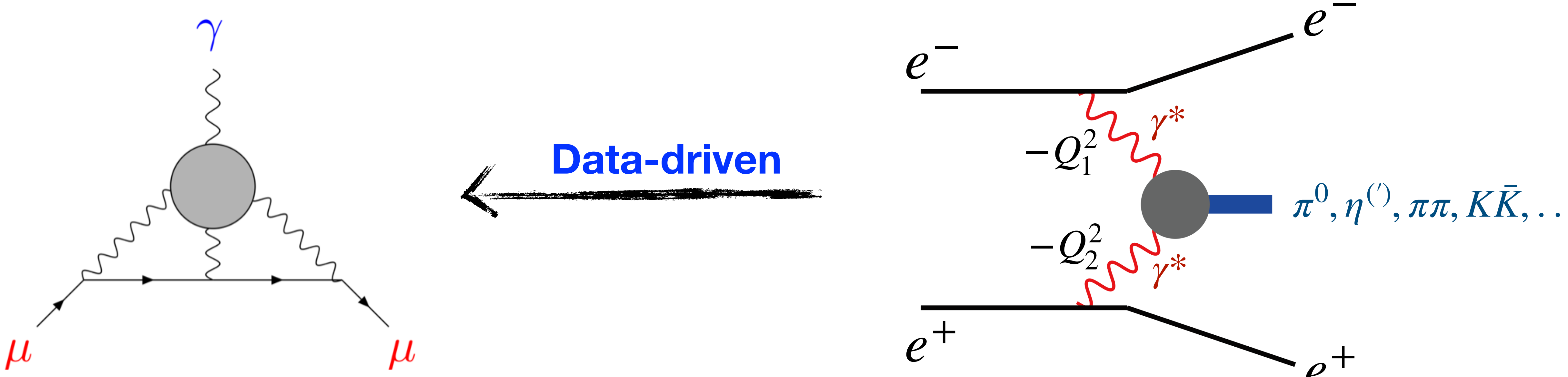
$\bar{\Pi}_i \propto$



- Experimental measurements of two-photon fusion

✓ $\gamma^*\gamma^* \to \pi^0, \eta, \eta'; \pi\pi, K\bar{K}, \pi\eta; 3\pi, 4\pi, \cdots$ at low-Q² virtualities

✓ Form factors and (p.w.) amplitudes



Current status of HLbL

- White paper T. Aoyama et al., Phys. Rept. 887, 1 (2020)

$$a_\mu^{\mathrm{HLbL}} = 92(19) \cdot 10^{-11}$$

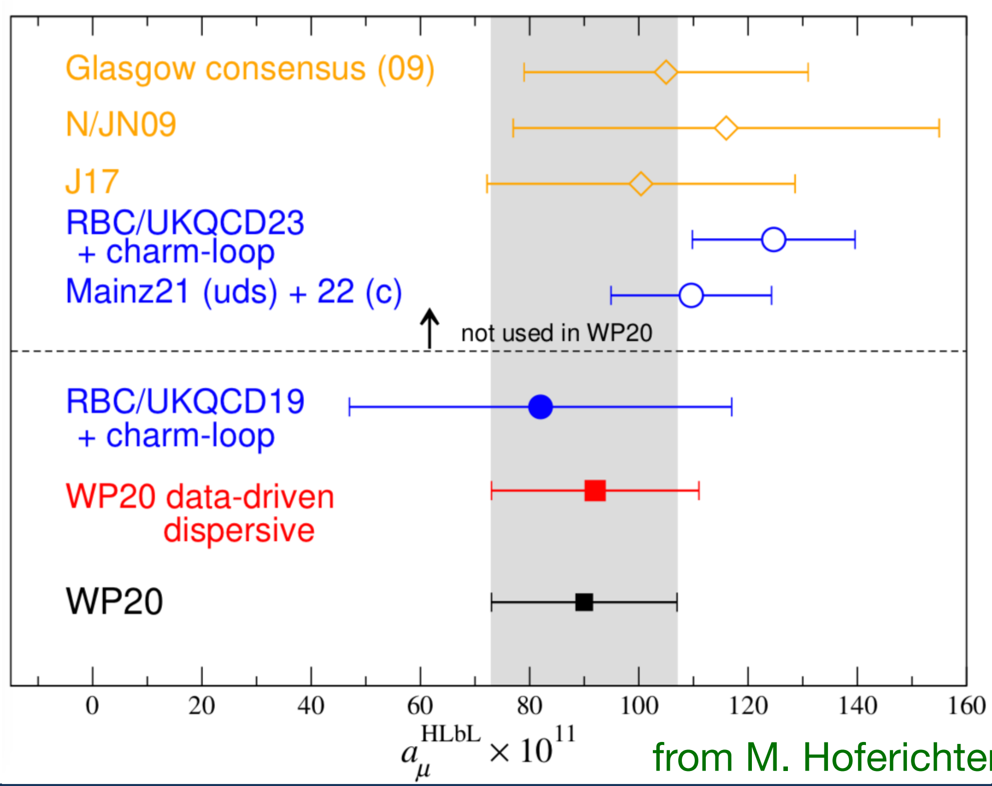
hadronic state	$a_\mu^{\mathrm{HLbL}}\ [10^{-11}]$
pseudoscalar poles	$93.8^{+4.0}_{-3.6}$
pion box	$-15.9(2)$
S-wave $\pi\pi$ rescatt.	$-8(1)$
kaon box	$-0.5(1)$

Well determined!

hadronic state	$a_\mu^{\mathrm{HLbL}}\ [10^{-11}]$
scalars+tensors $\gtrsim 1\,\mathrm{GeV}$	$\sim -1(3)$
axial vectors	$\sim 6(6)$
short distance	$\sim 15(10)$
heavy quarks	$\sim 3(1)$

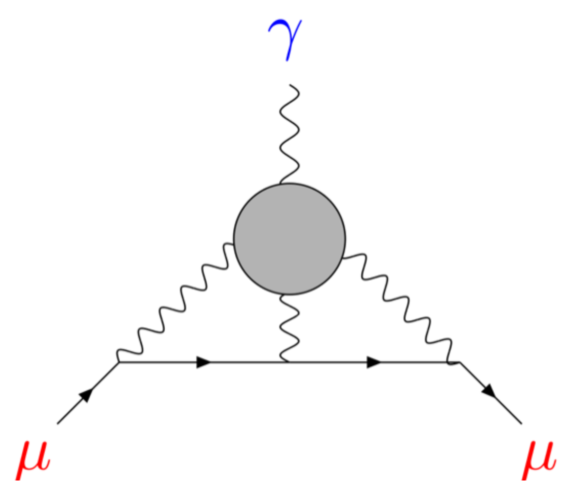
Major source of uncertainty

- Comparison with LQCD: uncontroversial

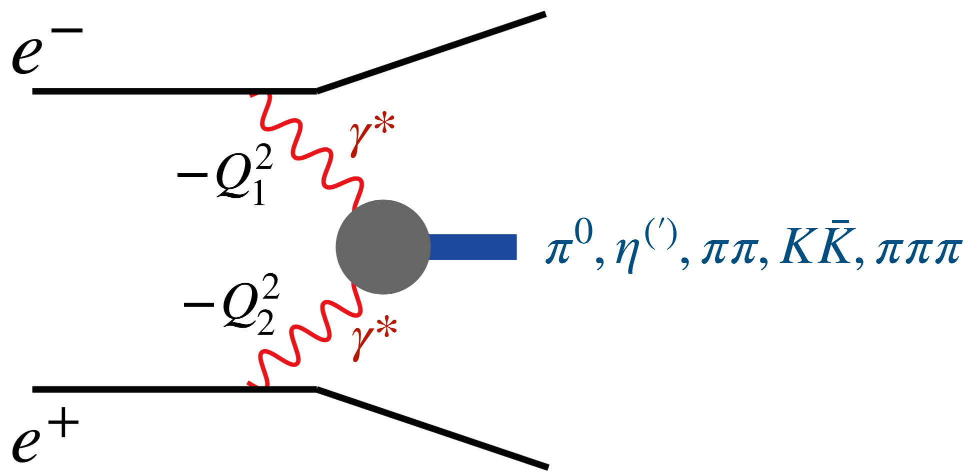


from M. Hoferichter's talk

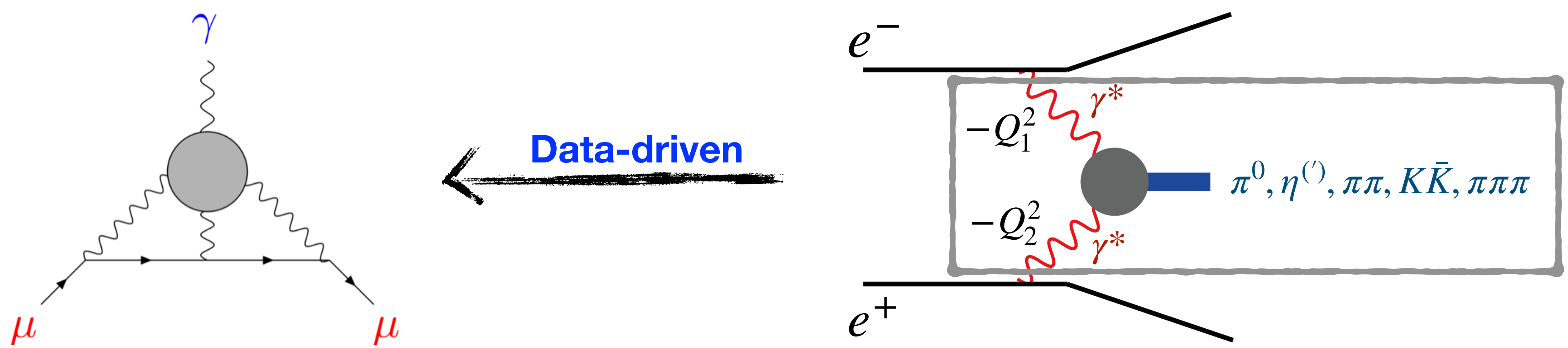
Needs better understanding of complicated hadronic dynamics to get reliable error estimate
—> 10% accuracy



Data-driven

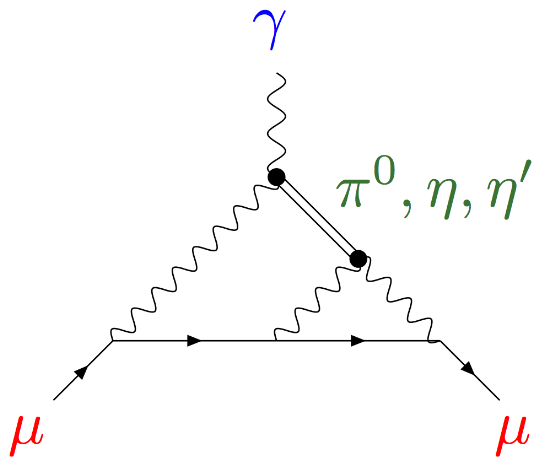


Photon-photon fusion



Photon-photon fusion

Two-photon fusion to pseudoscalars

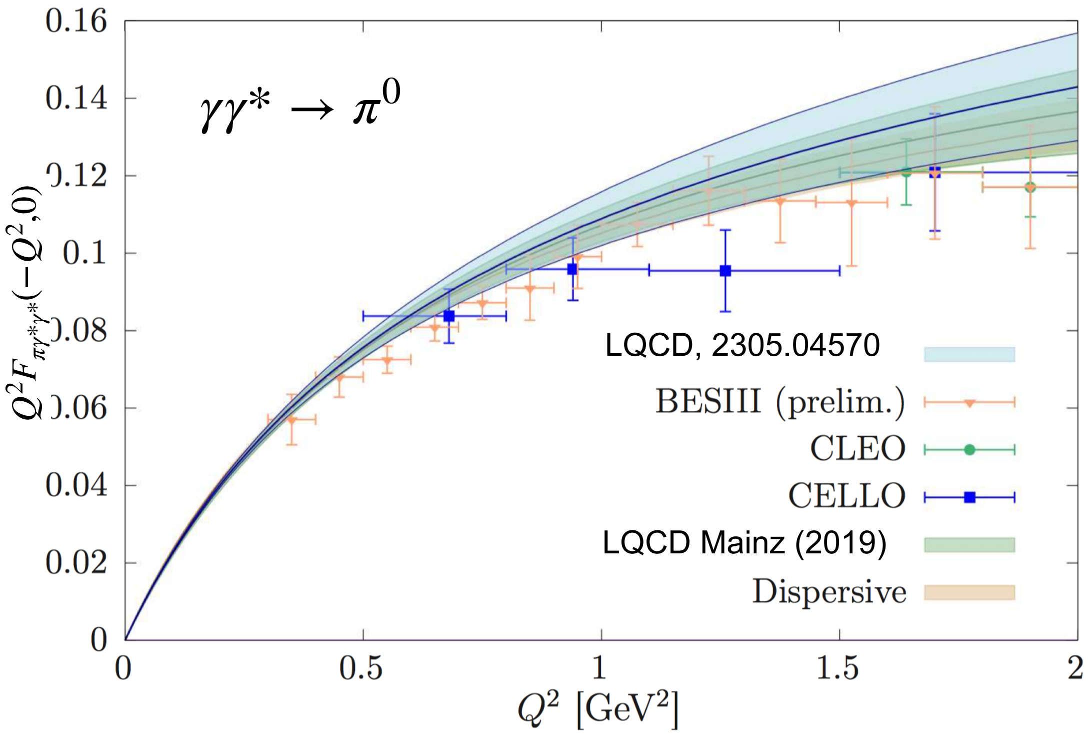


Jegerlehner & Nyffeler, PRD (2009)

$$a_\mu = \left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{+1} d\tau$$
$$\times \left\{ w_1(Q_1, Q_2, \tau)\, F_{\mathbf{P}}\left(-Q_1^2, -(Q_1+Q_2)^2\right) F_{\mathbf{P}}\left(-Q_2^2, 0\right) + w_2(Q_1, Q_2, \tau)\, F_{\mathbf{P}}\left(-Q_1^2, -Q_2^2\right) F_{\mathbf{P}}\left(-(Q_1+Q_2)^2, 0\right) \right\}$$

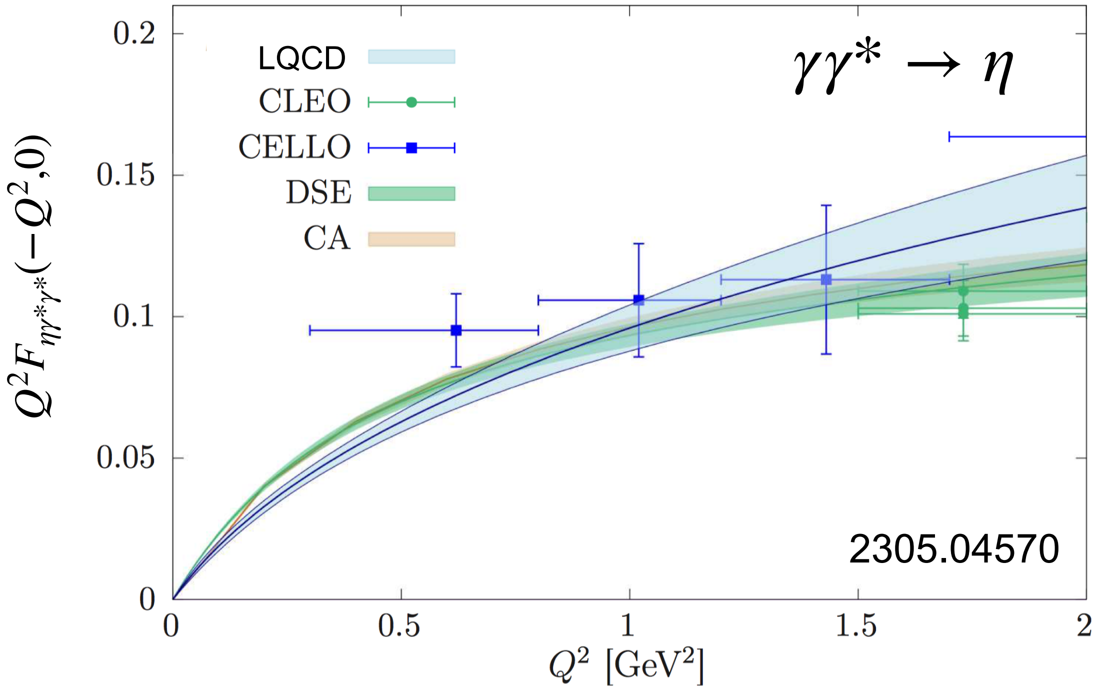
- Weight functions: w_1, w_2 are model independent!
  - ✓ Suppress large virtuality contributions
- **ONLY input:** single/double virtual transition form factors (TFF)

✓ π^0 TFF is well determined



✓ $\eta - \eta'$ mixing

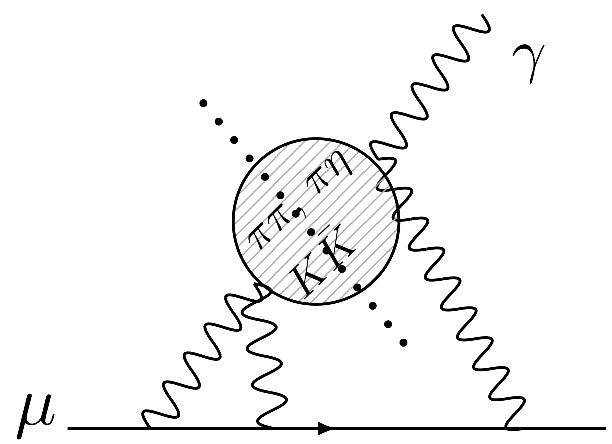
✓ No dispersive analysis available



→ Call for improvements for η, η′ TFFs

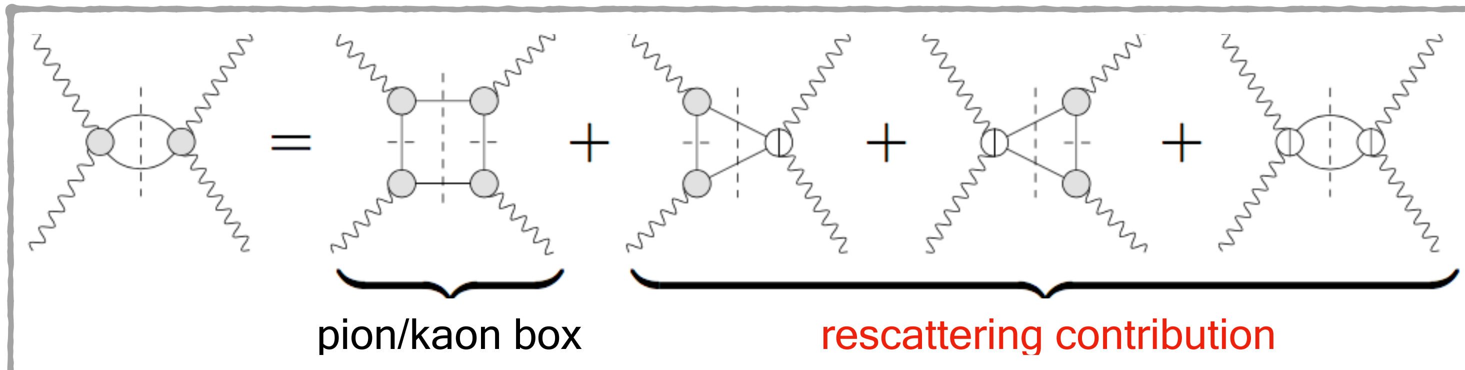
Two-photon fusion to $\pi\pi, \pi\eta, K\bar{K}$

Two-pseudoscalar contribution to a_μ^{HLbL}



Colangelo et al. (2014-2017)

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \sqrt{1-\tau^2} Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i\left(Q_1, Q_2, \tau\right) \bar{\Pi}_i\left(Q_1, Q_2, \tau\right)$$



- Expression of $\bar{\Pi}_i$ (s-wave) *G. Colangelo, et al., JHEP 04 (2017) 161, 1702.07347*

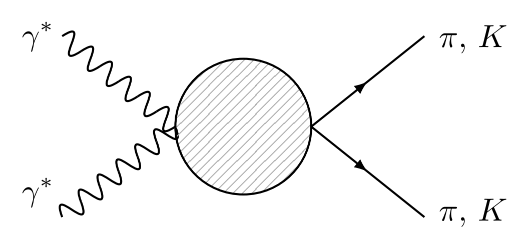
$$\bar{\Pi}_i^{J=0} \propto \frac{1}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{1}{\lambda_{12}(s')(s'-q_3^2)^2}\left[f(s')\,\mathrm{Im}h^0_{++,++}(s') - g(s')\mathrm{Im}h^0_{00,++}(s')\right]$$

Similar for the t, u-channels

✓ Helicity amplitudes of $\gamma^*\gamma^* \to \gamma^*\gamma^*$ can be obtained via $\gamma^*\gamma^* \to \pi\pi, K\bar{K}, \pi\eta$

Unitarity

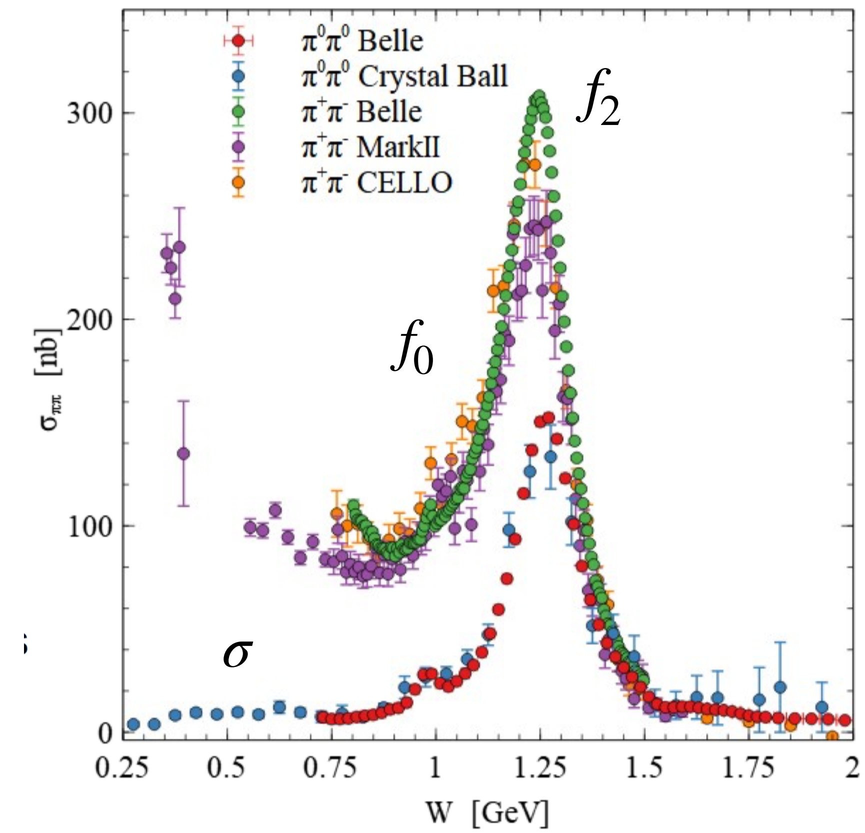
$$\mathrm{Im}_s^{\pi\pi}\, h^J_{\lambda_1\lambda_2,\lambda_3\lambda_4}(s) = \eta_i\eta_f \frac{\rho_\pi(s)}{16\pi S} h_{J,\lambda_1\lambda_2}(s) h^*_{J,\lambda_3\lambda_4}(s)$$



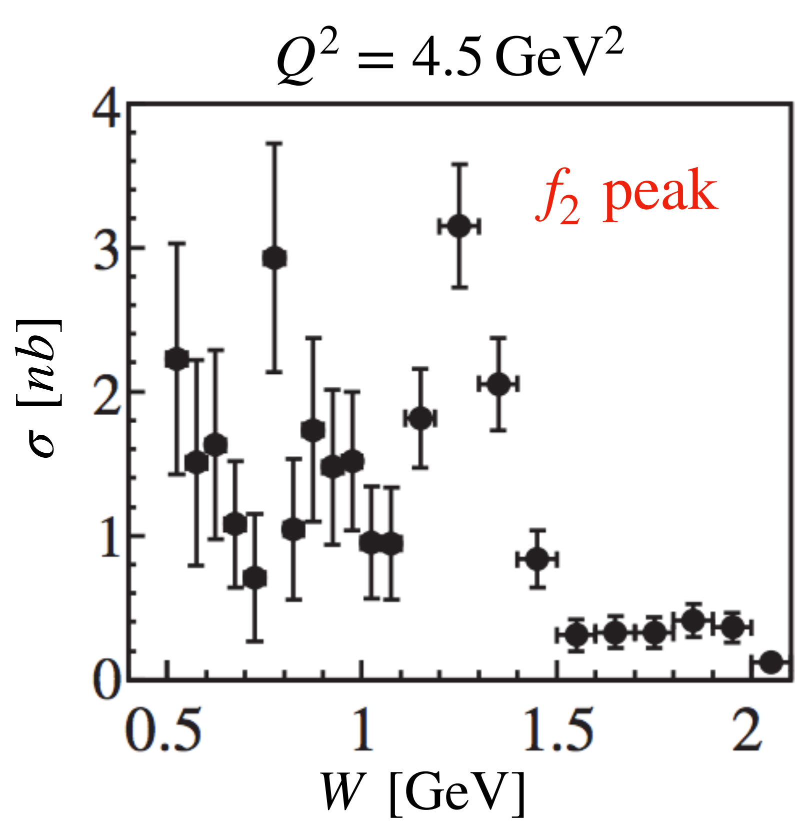
Inputs: p.w. amplitudes of $\gamma^*\gamma^* \to \pi\pi, K\bar{K}, \pi\eta\ldots$ at arbitrary virtualities

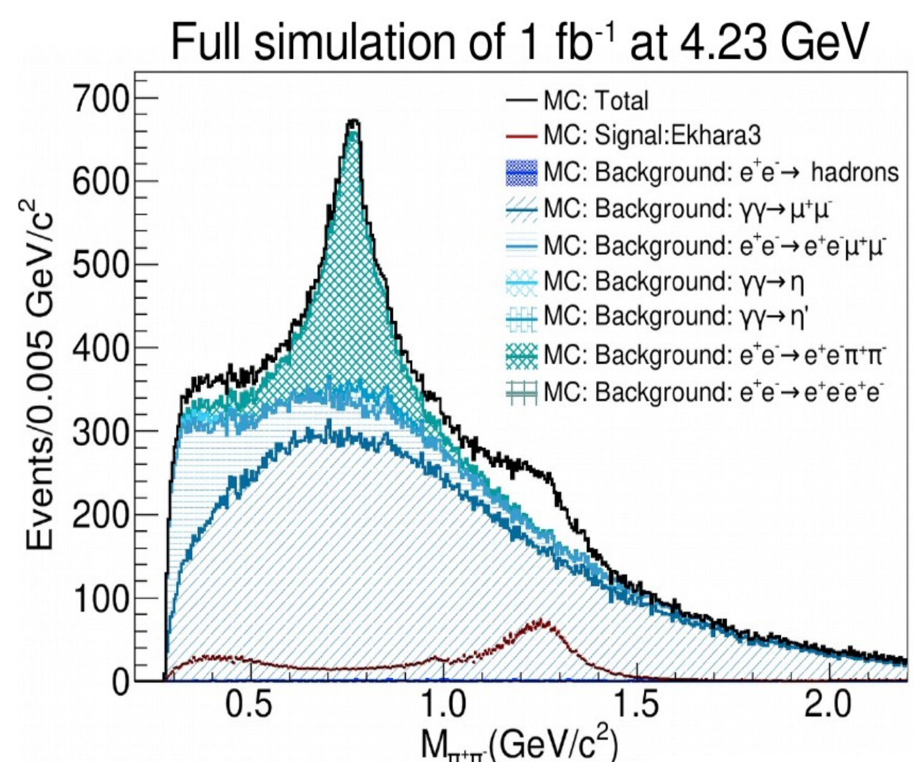
Expt. status for $\gamma^{(*)}\gamma^{(*)} \to \pi\pi, K\bar{K}, \ldots$

- Data for real photon fusion
  - MarkII('90), CELLO ('92), Crystal Ball ('90), Belle ('07 '09) ; BESIII prelim.
    - ▸ **M. Küßner's talk @ 23.06**
  - Access the scalar and tensor resonances
    - ✓ Partial wave analysis



- Data for single virtual process
  - Belle: $\gamma\gamma^* \to \pi^0\pi^0$ *PRD93(2016)032003*
  - BESIII: $\gamma\gamma^* \to \pi^+\pi^-$ on going analysis





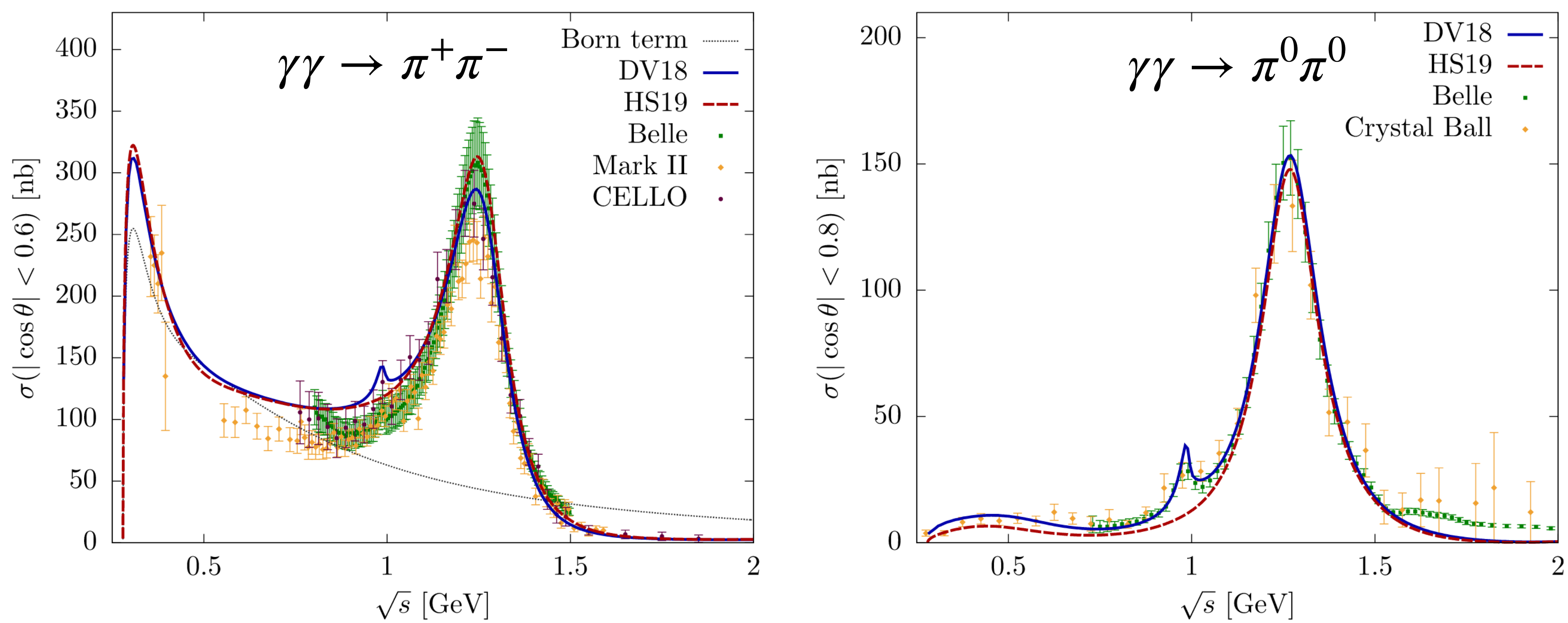
$0.2 \le Q^2 \le 2$

$|\cos\theta^*| \le 1$

J. Phys.: Conf. Ser. **1137** 012008 (2019)

Dispersive analysis for $\gamma^{(*)}\gamma^{(*)} \to \pi\pi, \ldots$

- Single channel: S-wave helicity amplitude $\gamma^*\gamma^* \to \pi\pi$
  - G. Colangelo, et al., JHEP 04 (2017) 161
- Extend to $K\bar{K}$ channel: S+D waves helicity amplitudes
  - I. Danilkin, M. Vanderhaeghen, PLB 789 (2019) 366
  - M. Hoferichter, P. Stoffer, JHEP 07 (2019) 073



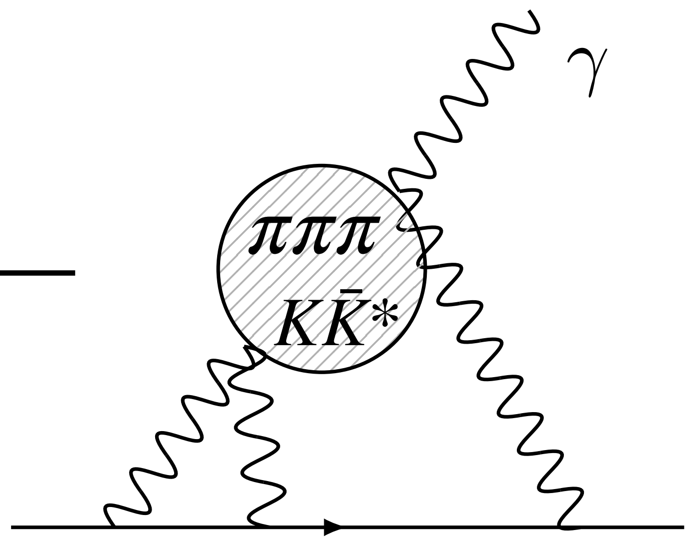
WP(2020)

→ Need to be validated by the upcoming BESIII $\gamma\gamma^* \to \pi^+\pi^-$ data!

- Couple channel: helicity amplitude $\gamma^{(*)}\gamma^{(*)} \to \pi\eta, K\bar{K}$
  - ▸ **Details can be seen in O. Deineka's talk @ 22.06**

Two-photon fusion to three pions

hadronic state	$a_\mu^{\mathrm{HLbL}}\,[10^{-11}]$
scalars+tensors $\gtrsim 1\,\mathrm{GeV}$	$\sim -1(3)$
axial vectors	$\sim 6(6)$
short distance	$\sim 15(10)$
heavy quarks	$\sim 3(1)$

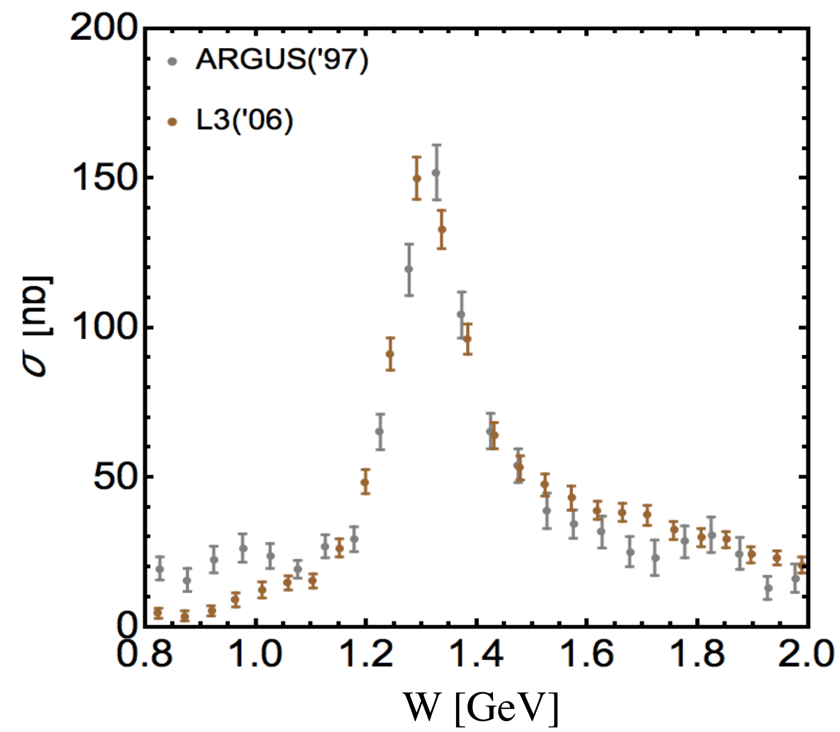


- Better control the uncertainty from the axial vectors and tensors contribution
  - ✓ $a_1(1260) \to \pi\pi\pi$
  - ✓ $f_1(1285) \to \eta\pi\pi(\sim 50\%),\ 4\pi(\sim 30\%)$
  - ✓ $f_1(1420) \to K\bar{K}^*\ (\sim 96\%)$
  - ✓ $a_2(1320) \to \pi\pi\pi\,(\sim 70\%)$
  - ✓ …
- Need to study $\gamma^{(*)}\gamma^{(*)} \to 3\pi, K\bar{K}^*, \ldots$

Two-photon fusion to three pions

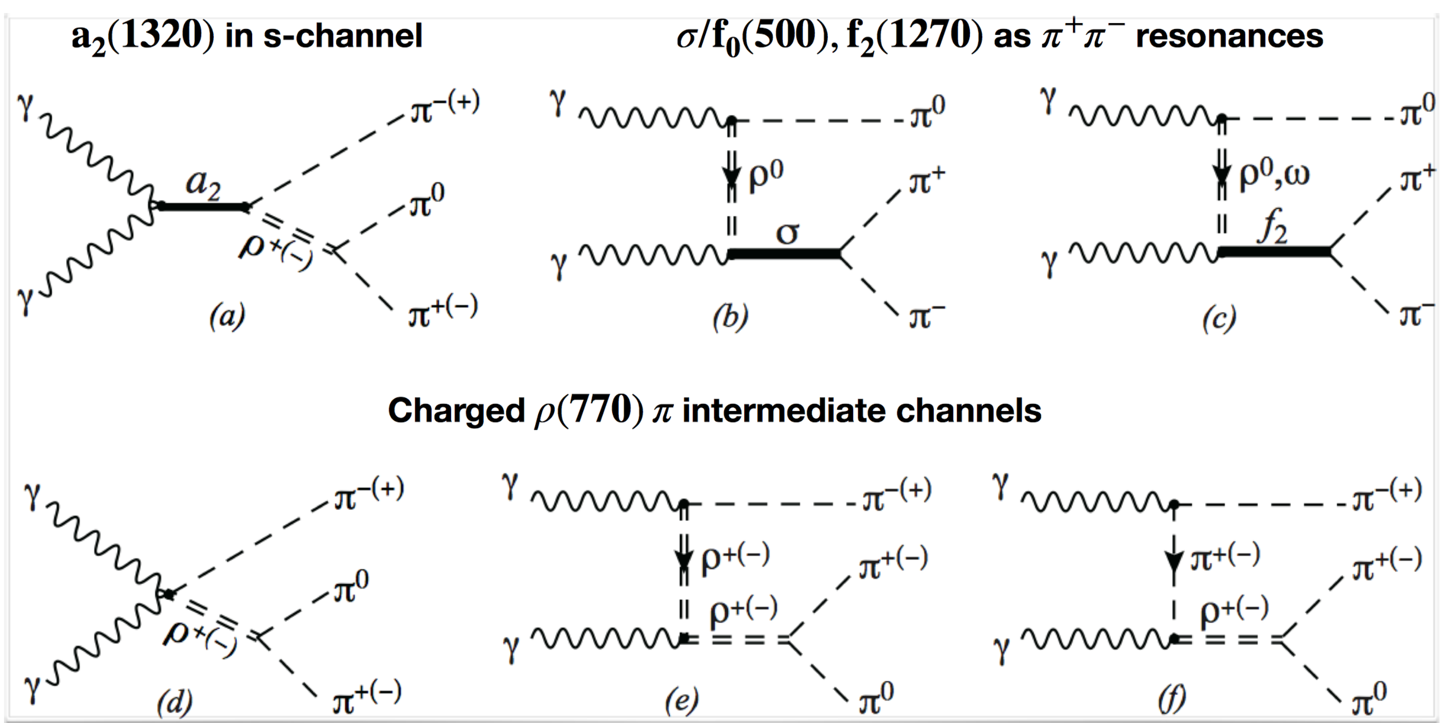
- Experimental data of $\gamma\gamma \to \pi^+\pi^-\pi^0$
  - ARGUS Z. Phys. C(1997), L3 PLB(1997), EPJA(2006)
  - Significant difference in low-energy region

    → Call for experimental validation:
    forthcoming BESIII data with high-statistics



- Theoretical studies (very limited)
  - Current Algebra, ChPT S. L. Adler, et al.,PRD(1971); P. Talavera, et al., PLB(1996)

- We proposed a phenomenological model for $\gamma\gamma \to \pi^+\pi^-\pi^0$

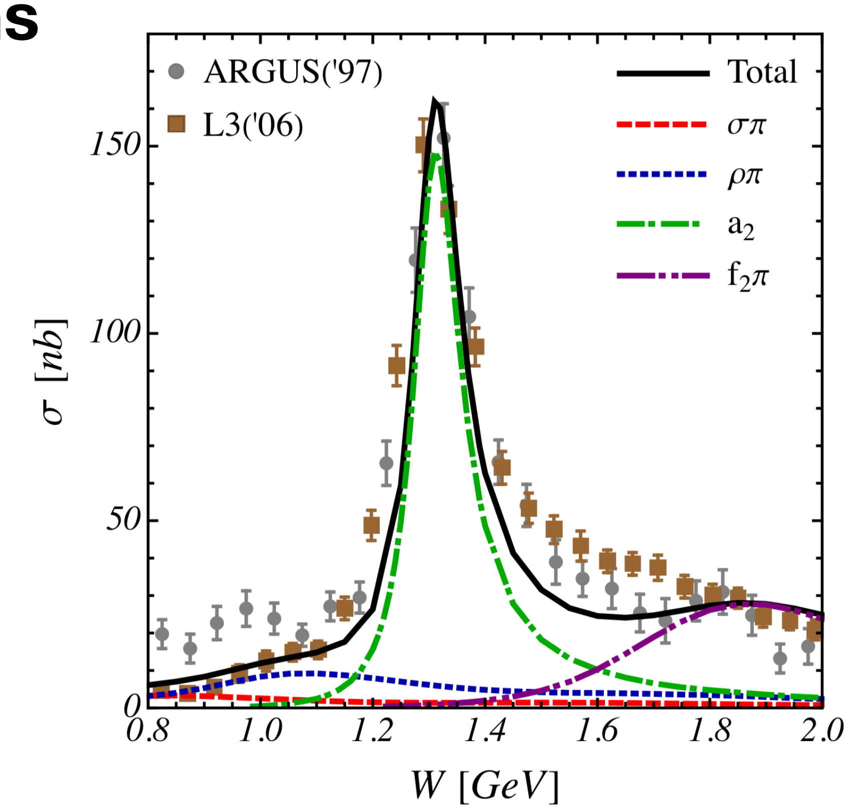


- Cover the low and the intermediate energy region
- Describe the experimental data of ARGUS and L3

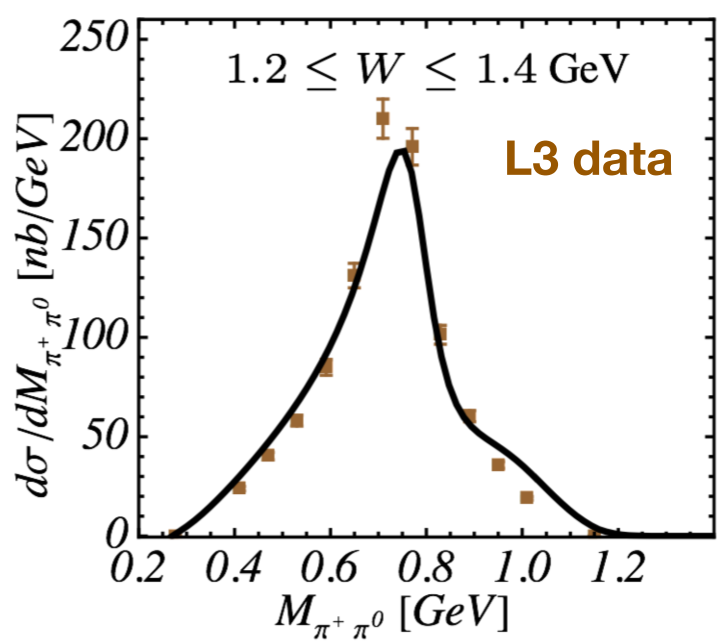
XLR, I. Danilkin, M. Vanderhaeghen, PRD107,054037(2023)

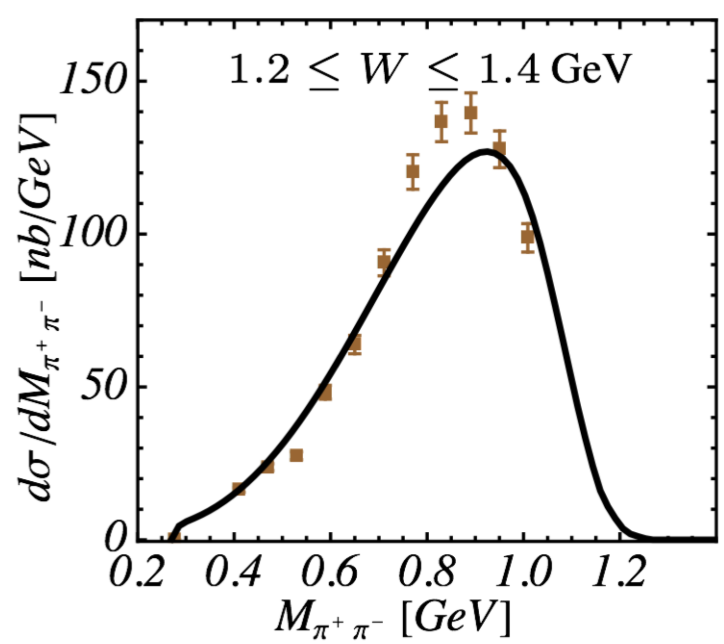
Description of Expt. data

- One fit parameter in our model
  - $g_{f_2\rho\gamma}$ is fitted by reproducing the total cross section at W=1.85 GeV
  - **Others are fixed via the corresponding decay widths**
- Total cross section
  - Our result is consistent with L3 data at low energies
  - a2(1320) production: the dominant contribution
  - $f_2(1270)\pi^0$ mechanism: good description of the total cross section



- Invariant mass distribution





XLR, I. Danilkin, M. Vanderhaeghen, PRD107,054037(2023)

Summary and outlook

- SM prediction for $(g-2)_\mu$: 4.2σ deviation from experimental value
  - Next release of FNAL(E989) is very soon!
  - High precision measurement is under construction at JPARC (E34)
  - → Precision of hadronic contribution: **HVP** $\sim 0.2\,\%$ **, HLbL** $\leq 10\,\%$
- HVP tensions: $e^+e^- \to \pi^+\pi^-$ data, data-driven vs. LQCD
  - **MUonE** expriment ▸ **M. Goncerz's talk @ 26.06**
- Data-driven HLbL
  - Improve the η, η' transition form factors
  - Expect the release of BESIII data: $\gamma\gamma^* \to \pi^+\pi^-$
  - Study the photon-photon fusion: $\gamma^{(*)}\gamma^{(*)} \to 3\pi, K\bar{K}^*, 4\pi \ldots$
    - ✓ Better control the axial vectors and tensors contributions

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  - Expect the release of BESIII data: $\gamma\gamma^* \to \pi^+\pi^-$
  - Study the photon-photon fusion: $\gamma^{(*)}\gamma^{(*)} \to 3\pi, K\bar{K}^*, 4\pi \dots$
    - ✓ Better control the axial vectors and tensors contributions

Thank you for your attention!

Backup slides

Sensitivisty of a_ℓ to short distance physics

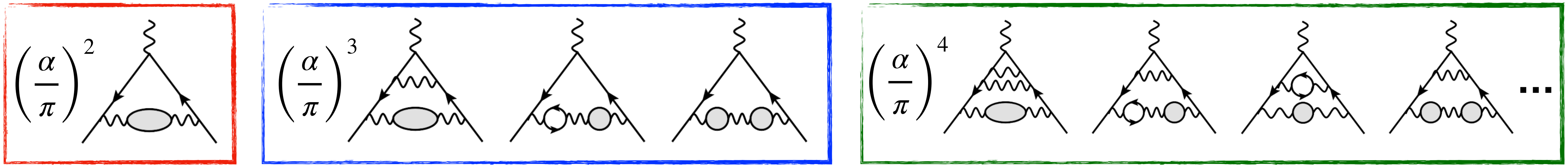
$$\frac{\delta a_\ell}{a_\ell} \sim \frac{m_\ell^2}{\Lambda^2}$$

where Λ is an UV cut–off characterizing the scale of new physics.

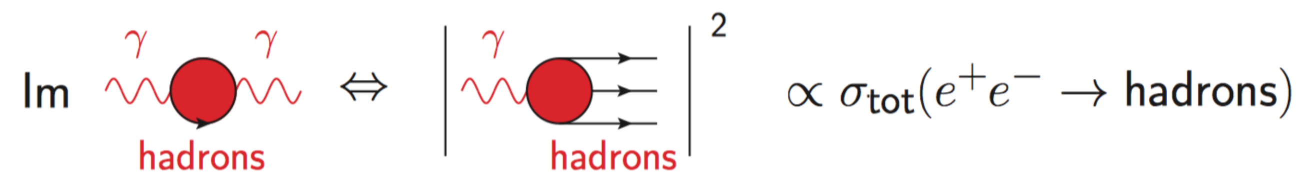
Berestetskii et al., Zh. Eksp. Teor. Fiz. 30 (1956) 788 [Sov. Phys. JETP 3 (1956) 761]

- The anomalous magnetic moment of leptons mediates helicity flip transitions.
  - Massive particle allows to have Helicity flips
  - The transition amplitude is proportional to the mass of the particle.

Beyond leading order HVP



- Data-driven approach
  - Based on analyticity and unitarity



  - **Dispersion integrals over cross section of e^+e^- annihilation**

$$a_\mu^{\text{LO-HVP}} = \frac{\alpha^2}{3\pi^2}\int_{M_\pi^2}^{\infty} ds\, \frac{K^{(1)}(s)}{s} R(s)$$

$$R(s) = \frac{\sigma^0\left(e^+e^- \to \text{hadrons}(+\gamma)\right)}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

J. Phys. Radium 22, 121 (1961)

High orders HVP: same R-ratio, different kernels *A.Kurz, et al., PLB 734(2014)144*

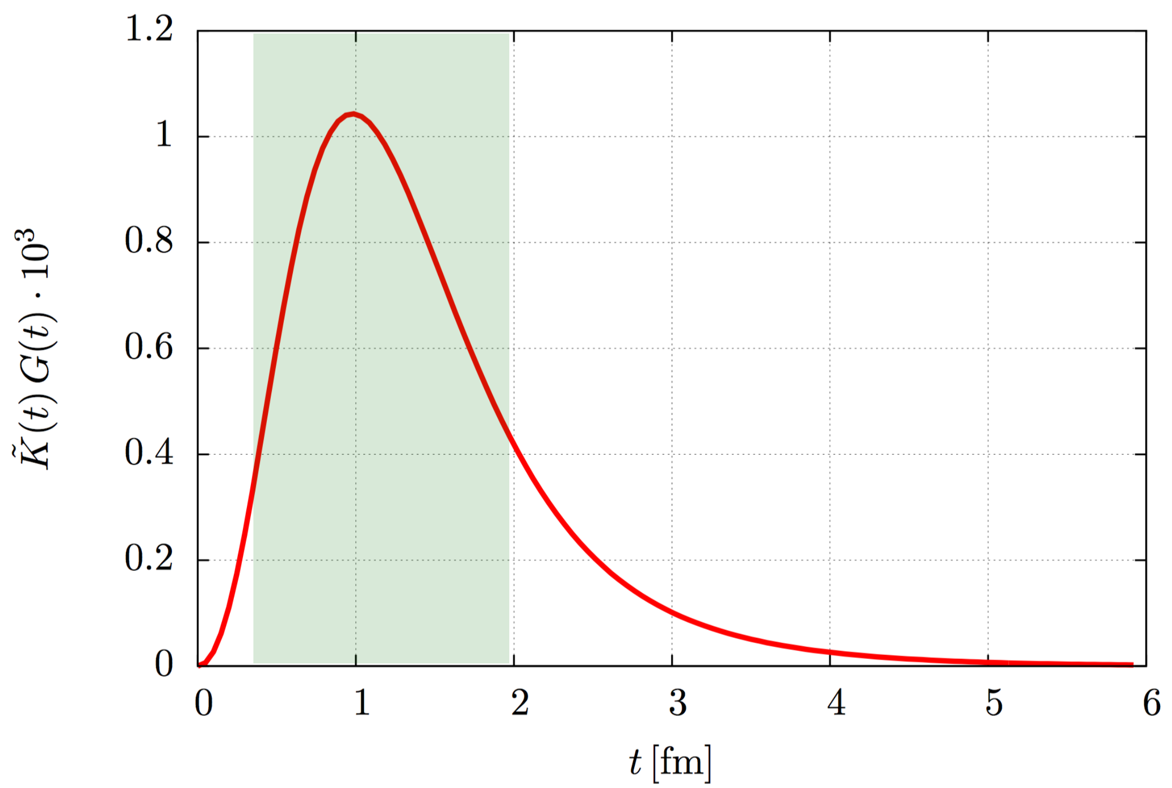
$$a_\mu^{\text{NLO-HVP}} = \frac{\alpha^3}{3\pi^3}\left[\int_{M_\pi^2}^{\infty} ds\, \frac{K^{(2)}(s)}{s} R(s) + \int_{M_\pi^2}^{\infty} ds\, ds'\, \frac{K^{(2)\prime}(s,s')}{s\, s'} R(s)\, R(s')\right]$$

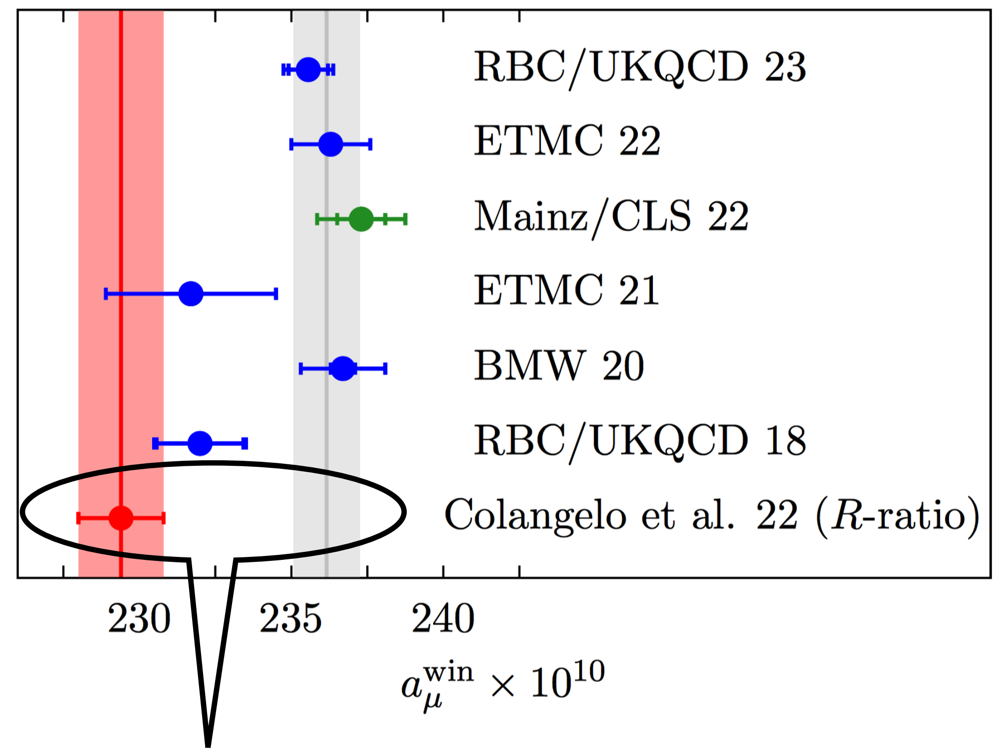
$$a_\mu^{\text{NNLO-HVP}} = \frac{\alpha^4}{3\pi^4}\left[\int_{M_\pi^2}^{\infty} ds\, \frac{K^{(3)}(s)}{s} R(s) + \int_{M_\pi^2}^{\infty} ds\, ds'\, \frac{K^{(3)\prime}(s,s')}{s\, s'} R(s)\, R(s') + \int_{M_\pi^2}^{\infty} ds\, ds'\, ds''\, \frac{K^{(3)\prime\prime}(s,s',s'')}{s\, s'\, s''} R(s)\, R(s')\, R(s'')\right]$$

- Intermediate window observable $a_\mu^{\text{HVP,win}}$

$$a_\mu^{\text{HVP,win}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt \tilde{K}(t) G(t) \boldsymbol{W(t; t_0, t_1)}$$

- Constrain the safe integral regions
  - ✓ Reduce FV and discretized effects
  - ✓ Better statistic precision



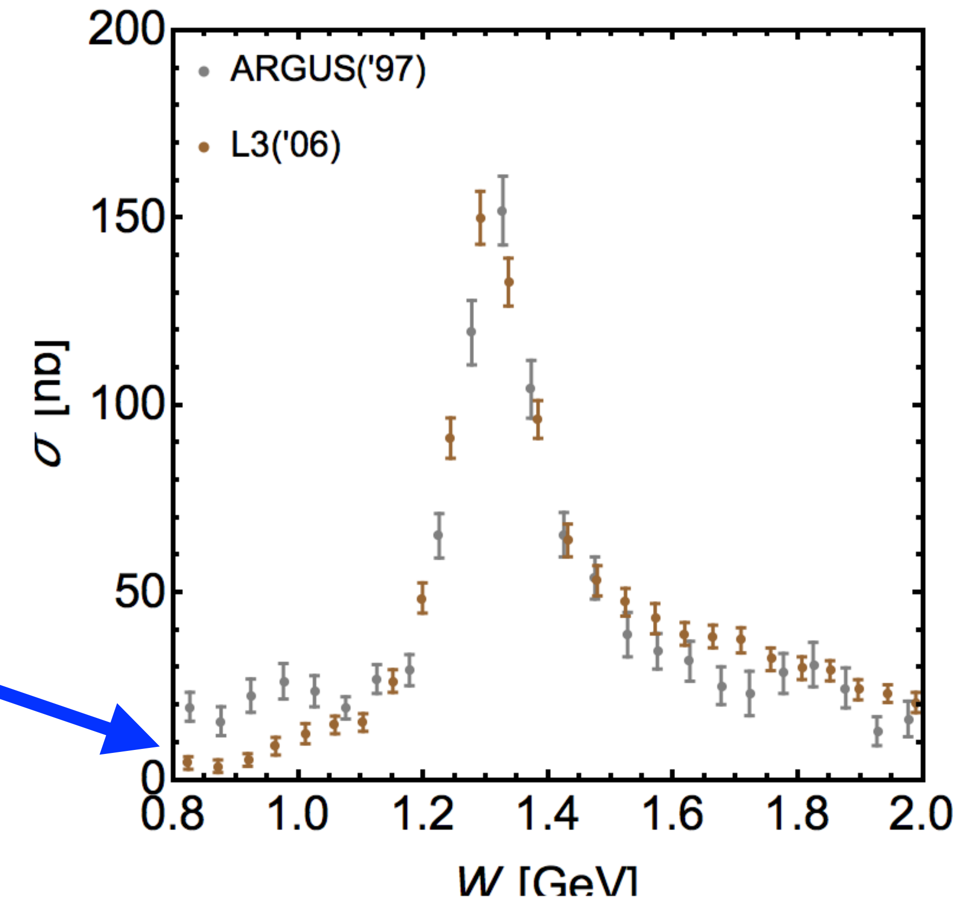


Tension $3.8\,\sigma$

H. Wittig 2306.04165

Two-photon fusion to three pions

- Experimental data of $\gamma\gamma \to \pi^+\pi^-\pi^0$
  - Data are rather old and have low statistics
    - ✓ ARGUS collab. *Z. Phys. C 74, 469 (1997)*
    - ✓ L3 collab. *PLB 413, 147(1997)*
      updated analysis *EPJA27,199(2006).*
  - Significant difference in low-energy region



- Theoretical studies (very limited)
  - Current Algebra and the linear sigma model
    S. L. Adler, et al.,PRD(1971); T.F.Wong,PRL(1971); R. Aviv,PRD(1972)
  - Chiral perturbation theory (ChPT) up to NLO
    J.W. Bos, PLB337, 152(1994); P. Talavera, et al., PLB376, 186(1996)
  - Those studies focused on the very low energies
    - ✓ nearby the 3π threshold of the two-photon fusion
    - ✓ $0.41 < W < 0.7$ GeV