$\pi\pi$-Scattering with IAM in the Finite Volume

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Introduction

Tasks to do:

- Current methods:
  - Volume-dependence
  - *LHC* is ignored at loops
  - $\tilde{V} = V$
- Generalization:
  - Volume- and Exponential-dependencies
  - Correct treatment of the *LHC*
  - $\tilde{V} = V + \Delta V$

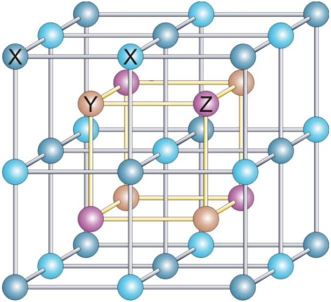


Figure: Lattice QCD

M. Luscher (DESY). Volume Dependence of the Energy Spectrum. Commun. Math. Phys. 105 (1986) 153-188.

Consequence of the Discretization

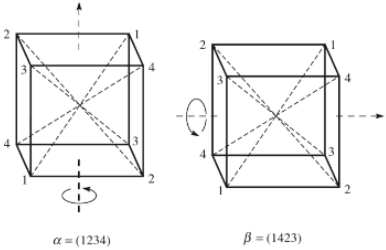


Figure: Cube rotations

- The $SO(3)$-symmetry is broken (3 generators) $\rightarrow O_h$ (24 generators)
- Lorentz-symmetry is broken $\rightarrow$ PV is not applicable.
- Volume- and Exponential-dependence on amplitudes.
- Discretization of the momentum $\rightarrow$ Shell-labeling
- $T(p, p') \rightarrow T(p.p', p', p)$

Inverse Amplitude Method (IAM)

For elastic scattering, every t partially projected, an amplitude:

$$t = t_2 + t_4 + \ldots$$

must fulfill: $Im\, t^{-1} = \sigma |t|^2$ ($Im\, t_2 = 0$, $Im\, t_4 = \sigma t_2^2$). Then for a given $F(s) = \frac{t_2^2}{t}$, we have:

$$F(s) = F(0) + F'(s)s + \frac{1}{2}F''(0)s^2 + \frac{s^3}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{Im\, F(s')}{s'^3(s'-s)} + LC(F) + PC$$

The perturbative conditions imply:

$$\frac{t_2^2}{t} = F = t_2 - t_4$$

then,

$$t_{IAM}(s) = \frac{t_2^2}{t_2 - t_4}$$

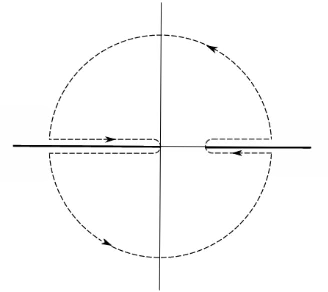


Figure: Contour of integration in the complex s-plane

ArXiv: 1205.35082v1 [hep-lat] (2012)

Inverse Amplitude Method (IAM)

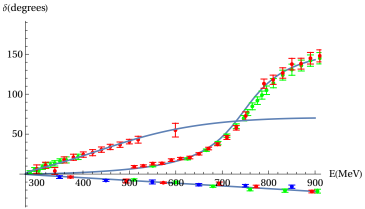


Figure: δ-shift for $I = \{0, 1, 2\}$ at different energies.

From

$$\begin{aligned} V &= \frac{t_2^2}{t_2 - t_4^{tree}} \\ t_4 &= t_4^{tree} + t_2^2 J(s) \end{aligned}$$

we get

$$T = V + VJT \tag{1}$$

Phys. Rev D 59, 074001 (2007). J.A. Oller. E. Oset, and J. R. Peláez.

Finite Volume $\pi\pi$-Scattering

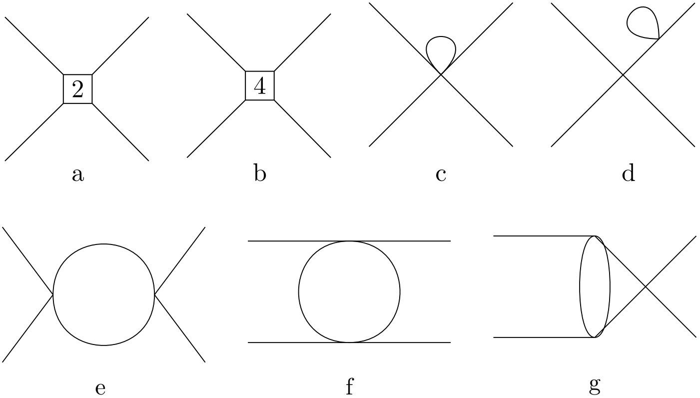


Figure: Feynman diagrams for $\pi\pi$ scattering amplitude up to fourth order in ChPT.

Finite Volume $\pi\pi$-Scattering

Amplitudes

$$T = \underbrace{T_2 + T_4 + \sum_{i=\{s,t,u\}} c_i \bar{J}(i)}_{T_4^{(loop)}} = T_\infty + c_s \bar{J}(s)$$

$$\tilde{T} = \underbrace{T_\infty + c_s \bar{J}(s)}_{T} + \underbrace{\sum_{i=\{H,s,t,u,t^2,u^2\}} c_i DJ_i}_{DT} = T_\infty + DT + c_s \tilde{J}$$

$$\mathcal{T}(E, L, \hat{p}, \hat{p}') = \frac{T_2^2}{DT + c(E, \hat{p}, \hat{p}')\left(\tilde{J}(E, L) - \bar{J}(E)\right)}$$

Phys. Rev. D, vol 65, 054009 (2002). A. Gómez Nicola and J. R. Peláez

Phys. Rev. D 73, 074501 (2006). P.F. Bedaque, I. Sato and A. Walker-Loud

Dynamical equation and the Universal Function

IAM

$$\tilde{\mathcal{T}} = \frac{\tilde{V}}{1 - \underbrace{\frac{2\tilde{V}c_s}{\sqrt{s}T_2^2}}_{\tau} \mathcal{S}(s)}$$

where $\tilde{V} = \frac{T_2^2}{T_2 - T_4 - DT}$, and

$\mathcal{S} = -\frac{1}{8\pi^2} \lim_{q_{\text{max}} \to \infty} \left(\frac{1}{2L} \sum_{\vec{n} \le q_{\text{max}}} \frac{1}{\vec{n}^2 - \frac{k_{\text{cm}}^2 L^2}{4\pi^2}} - q_{\text{max}} \right)$

Quantum condition:

$$1 - \tau \mathcal{S}(s) = 0$$

Dynamical equation:

$$\tilde{\mathcal{T}} = \tilde{V} + \tau \mathcal{S}(s) \tilde{\mathcal{T}}$$

Phys. Rev. D 73, 074501 (2006). P.F. Bedaque, I. Sato and A. Walker-Loud

Green Functions and Differences

The loop sum-integrals can be defined as follows:

$$\tilde{H} = \sum\!\!\!\!\!\!\int \frac{1}{q^2 - m^2}$$

$$\tilde{J}(Q) = \sum\!\!\!\!\!\!\int \frac{1}{(q-Q)^2 - m^2}$$

$$\tilde{J}_{2Q}(Q) = \sum\!\!\!\!\!\!\int \frac{q_4^2}{q^2 - m^2} \frac{1}{(q-Q)^2 - m^2}$$

We should note that our discretization is only spatial, meaning that we divide the space into cubes, while the temporal component remains continuous.
We define the difference between the finite and infinite volume as follows:

$$Df = \int \frac{dq_0}{2\pi i} \left[\frac{1}{L^3} \sum_{\vec{q} = \frac{2\pi\vec{n}}{L}} - \int \frac{d^3 q}{(2\pi)^3} \right] f(\vec{q})$$

DJ_H Computation

The tadpole integral contributes to modifications in m_π and f_π, resulting in the following volume corrections:

$$
\begin{aligned}
DJ_H &= \int \frac{dq_0}{2\pi i}\left[\frac{1}{L^3}\sum_{\vec{q}=\frac{2\pi\vec{n}}{L}} - \int \frac{d^3q}{(2\pi)^3}\right]\frac{i}{q^2-m^2} \\
&= \left[\frac{1}{L^3}\sum_{\vec{q}=\frac{2\pi\vec{n}}{L}} - \int \frac{d^3q}{(2\pi)^3}\right]\frac{1}{2\omega_q} \\
&= \frac{m}{4\pi^2 L}\sum_n \frac{O_n}{n} K_1(nmL) \\
&= \frac{m^2}{16\pi^2}\int_0^\infty x^2 e^{-x}\left(\vartheta_3\left(0; e^{-\frac{m_\pi^2 L^2}{4x}}\right)^3 - 1\right)
\end{aligned}
$$

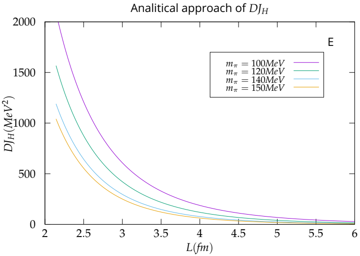
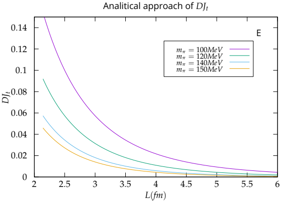
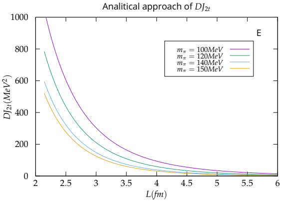


Figure: Comparison between analytic and numeric DJ_H

t- and u-Differences ($DJ_{u,t}$, $DJ_{2u,2t}$)



(a) Comparison between analytic and numeric $DJ_{t,u}$ at BSW's approach.



(b) Comparison between analytic and numeric $DJ_{2t,2u}$ at BSW's approach.

The integral/sum $J_{2t,2u}$ is given by:

$$DJ_{2t,2u} = \left[\frac{1}{L^3}\sum_{\vec{q}=\frac{2\pi\vec{n}}{L}} - \int\frac{d^3q}{(2\pi)^3}\right]\frac{1}{2\left(\omega_q + \omega_{q-Q}\right)}$$

where $\vec{Q} = \vec{T}, \vec{U}$. In the BSW limit, $DJ_{2u,2t} = \frac{1}{2}DJ_H$. With respect to $DJ_{u,t}$, we have:

$$DJ_{t,u} = \left[\frac{1}{L^3}\sum_{\vec{q}=\frac{2\pi\vec{n}}{L}} - \int\frac{d^3q}{(2\pi)^3}\right]\frac{1}{2\omega_q\omega_{q-Q}\left(\omega_q + \omega_{q-Q}\right)}$$

Indeed, it is straightforward to observe that $DJ_{u,t} = -\frac{1}{2}\frac{\partial}{\partial m}DJ_H$.

s-Difference(DJ_s)

$$DJ_s = \underbrace{\frac{2}{\sqrt{s}}\mathcal{S}}_{DJ_{s,\text{div}}} + DJ_{s,\text{conv}}$$

where

$$\mathcal{S}(s) = \sum_{n=0}^{q_{\max}} \frac{\mathbf{O}_n}{L^3}\frac{1}{4\omega_n^2 - s} - \frac{q_{\max} - \frac{\sqrt{4m_\pi^2 - s}}{2}\arctan\frac{2q_{\max}}{\sqrt{4m_\pi^2 - s}}}{8\pi^2}$$

$$DJ_{s,\text{conv}} = \sum_{\vec{n}\neq 0}^{\infty}\int \frac{d^3p}{(2\pi)^2} e^{iL\vec{q}\cdot\vec{n}}\left(-\frac{1}{\sqrt{s}}\frac{1}{\omega_q(2\omega_q + \sqrt{s})}\right)$$

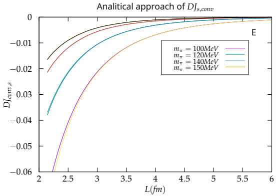


Figure: Comparison between analytic and numeric $DJ_{s,\text{conv}}$ at BSW's approach.

Exponential Volume Corrections

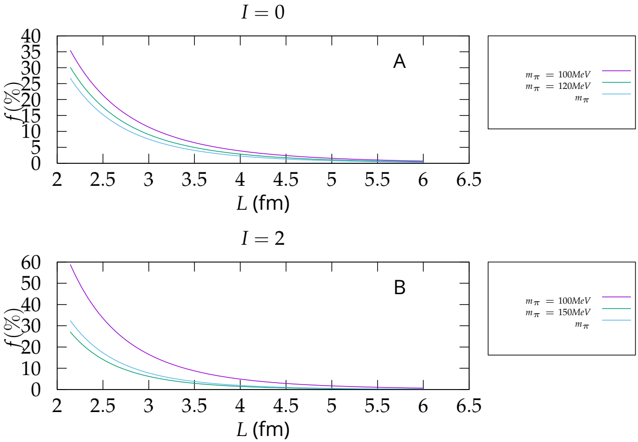


Figure: Where, $f = \frac{|\Delta k_{cm} \cot \delta|}{k_{cm} \cot \delta}(\%)$. Which represents the comparison between analytic and numeric results in the BSW approach.

Spectrum in a Finite Volume

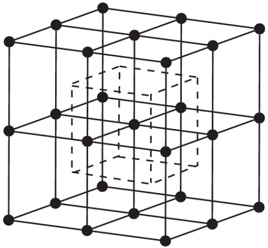


Figure: Cubic Lattice.

arXiv:1206.4141v2 [hep-lat] (2012)

arXiv: 0806.4495v2 [hep-lat] (2012)

Phys. Rev. D 97, 114508 (2018).

The symmetry group G of the cubic lattice. The irreps of the octahedral group of 24 elements (pure rotations) are:

- A_1: trivial one-dimensional rotation.
- A_2: one-dimensional representation, which assigns -1 to the conjugacy classes: $6C_4$ and $6C_2'$.
- E : two-dimensional rotations.
- T_1: three-dimensional rotations. $T_{\sigma\rho} = \cos(\omega_a)\delta_{\sigma\rho} + (1 - \cos\omega_a)n_\sigma^{(a)} n_\rho^{(a)} - \sin\omega_a \epsilon_{\sigma\rho\lambda} n_\lambda^{(a)}$
- T_2: three-dimensional rotations, which assigns -1 to the conjugacy classes: $6C_4$ and $6C_2'$.

The shells are the given surfaces where: $n_x^2 + n_y^2 + n_z^2 = \left|\frac{\vec{p}L}{2\pi}\right|^2$ and $\vec{p} = g\vec{p}_0$.

Alternatives of Expansion

$$\tilde{T}(p,p') = \tilde{V}(p,p') + \sum_k \underbrace{\left(\frac{2c\tilde{V}}{\sqrt{s}A_2^2}\right)}_{\tau(p,k)} \mathcal{S}(k)\tilde{T}(k,p') \qquad (2)$$

An arbitrary function $f(\vec{p})$ can be characterized by the shell of the momentum $\vec{p}$ belongs to and the orientation. So, the expansion over the cubic lattice is given by,

$$f(\vec{p}) = f(g\vec{p}_0) = \sum_{\Gamma}\sum_{\rho\sigma} T^{\Gamma}_{\rho\sigma}(g) f^{\Gamma}_{\sigma\rho}(\vec{p}_0)$$

The quantization condition is given by,

$$\det\left(\delta_{ss'}\delta_{\delta\sigma} - \frac{\mathbf{O}_s\mathcal{S}(s)}{G}\tilde{\tau}^{\Gamma}_{\delta\sigma}(s,s')\right) = 0$$

Phys. Rev. D 97, 114508 (2018).

The expansion in cubic harmonic (CH) basis is given by,

$$f^s(\hat{p}_j) = \sqrt{4\pi}\sum_{\Gamma\alpha}\sum_u f_u^{\Gamma\alpha s}\chi_u^{\Gamma us}(\hat{p}_j)$$

The quantization condition is given by,

$$\det\left(\delta_{uu'}\delta_{ss'} - \mathbf{O}_s\mathcal{S}(s)\tau^{\Gamma}_{su;s'u'}\right) = 0$$

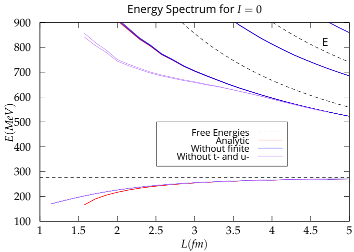


Figure: Energy levels for $I = 0$.

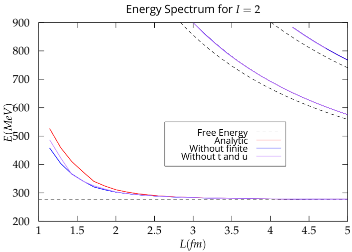


Figure: Energy levels for $I = 2$.

Conclusions and perspectives

Conclusions:

- The inclusion of the $t-$ and $u-$channel loop is useful for the correct computation of the energy levels.
- The finite volume effects are stronger in Isospin $I = 2$ then $I = 0$.
- The connection between T and $\tilde{T}$ is meadiated by $\tilde{J}(E, L)$ and the finite amplitud, DT.

Perpectives:

- To expand the formalism of CH and Irreps to the moving frame case.
- To study the dependence of the spectrum with the masses


That's all Folks!

Bedaque-Sato-Walker-Loud (BSW) Approach

Using the K-matrix, defined as:

$$\frac{1}{K(s,t,u)} = \frac{p_{\text{cm}} \cot \delta(s,t,u)}{16\pi\sqrt{s}} = V(s,t,u)$$

In this case, we can reexpress as follows:

$$\begin{aligned}\tilde{T} &= \frac{1}{\frac{1}{K(s,t,u)} - \frac{\Delta A}{T_2^2} - \frac{2c(s,t,u)}{\sqrt{s}A_2^2}\mathcal{S}(s)} \\ &= \frac{16\pi\sqrt{s}}{p_{\text{cm}} \cot \delta - \frac{16\pi\sqrt{s}\Delta A}{A_2^2} - \frac{32\pi c(s,t,u)}{A_2^2}\mathcal{S}(s)}\end{aligned}$$

$\tilde{T}$ represents a modified version of the Lüscher relation that includes finite volume corrections. In the BSW approach, where $s \to 4m_\pi^2$, the correction can be expressed as $\Delta(p_{\text{cm}} \cot \delta) = -32\pi m_\pi \frac{\Delta A(4m_\pi^2)}{A_2^2(4m_\pi^2)}$.