Derivative expansions of hadronic potentials coupled to quarks for $X(3872)$

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This talk is based on Ref. [I. Terashima and T. Hyodo, arXiv:2305.10689 [hep-ph]].

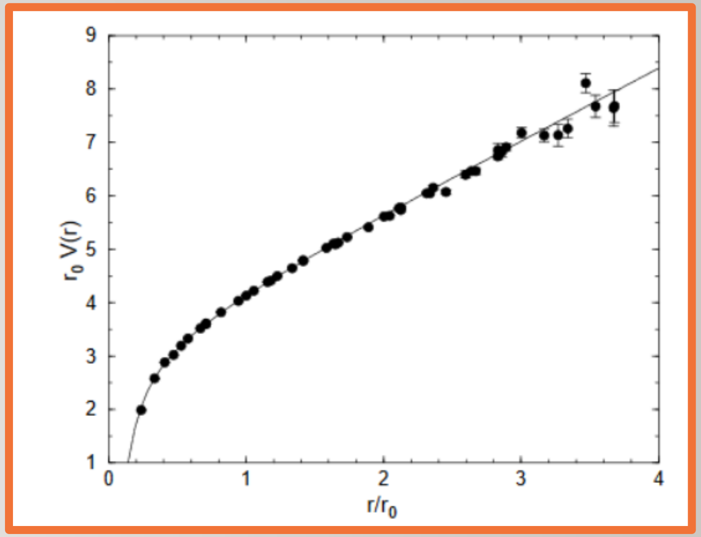
(Under peer review by Physical Review C)

Introduction

- Inter-quark and inter-hadron potentials

$$V^q(r) = -\frac{A}{r} + \sigma r + V_0 \underset{r\to\infty}{\rightarrow} \infty$$: Confinement potential

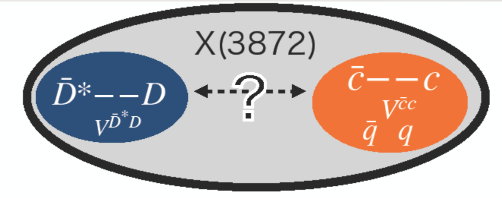
$$V^h(r) = g\frac{exp[-\mu r]}{r} + \cdots \underset{r\to\infty}{\rightarrow} 0$$: Scattering potential



CP-PACS, A. Ali Khan, et al., Phys. Rev. D 65, 054505 (2002)

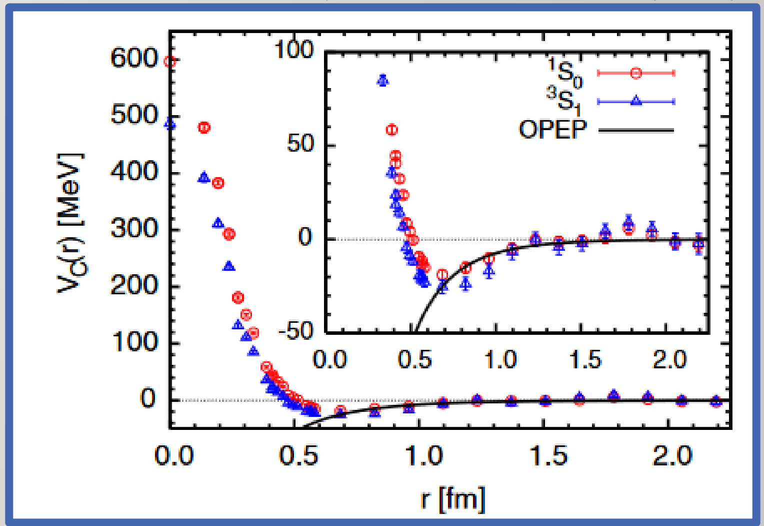
➢ $V^q(r)$ and $V^h(r)$ are calculated independently

◆ Mixed with same quantum numbers



➢ How about a channel coupling between **quark** and **hadron** degrees of freedom like $\mathbf{X(3872)}$?

A. Hosaka et al., PTEP **2016** (2016)



N. Ishii, S. Aoki, and T. Hatsuda, Phys. Rev. Lett. 99, 022001 (2007)

Flow of this talk

pp. 4-5
- Formulation of the channel coupling
- Showing the analytical results

Obtaining the effective potentials as **non-local** type

p.6
- Construction the model of $X(3872)$

pp. 7-8
- Formulation of the local potential from non-local potential with 2 methods

pp. 9-12
- Application to $X(3872)$
- Studying the numerical results

Channel coupling

✓ Formulation according to Feshbach method [H. Feshbach, Ann. Phys. 5, 357 (1958); ibid., 19, 287 (1962)]

- Hamiltonian H with channel between quark potential V^q and hadron V^h

$$H = \begin{pmatrix} T^q & 0 \\ 0 & T^h + \Delta \end{pmatrix} + \begin{pmatrix} V^q & V^t \\ V^t & V^h \end{pmatrix}$$

T^q, T^h:Kinetic energy

Δ:Threshold energy

V^t:Transition potential

- Schrödinger equation with wave functions of quark and hadron channels $|q\rangle, |h\rangle$

$$H \begin{pmatrix} |q\rangle \\ |h\rangle \end{pmatrix} = E \begin{pmatrix} |q\rangle \\ |h\rangle \end{pmatrix}$$

➢ Two set of equations with quark and hadron channels are obtained

Effective potential

- Eliminate quark channel to obtain an effective hamiltonian of hadron channel $H^h_{\text{eff}}(E)$

with, $H^h_{\text{eff}}(E)\,|h\rangle = E\,|h\rangle$, $V^h_{\text{eff}}(E)$

$$H^h_{\text{eff}}(E) = T^h + \Delta^h + \boxed{V^h + \boxed{V^t G^q(E) V^t}}$$

✓ No approximation

✓ G_q is the Green function of quark channel

$$G_q(E) = (E - (T^q + V^q))^{-1}$$

➢ Quark channel contribution by coupled channels

- Coordinate representation with initial relative coordinate $\boldsymbol{r}$ and final $\boldsymbol{r}'$

$$\langle \boldsymbol{r}'_h \mid V^h_{\text{eff}}(E) \mid \boldsymbol{r}_h \rangle = \langle \boldsymbol{r}'_h \mid V^h \mid \boldsymbol{r}_h \rangle + \boxed{\sum_n \frac{\langle \boldsymbol{r}'_h \mid V^t \mid \phi_n \rangle \langle \phi_n \mid V^t \mid \boldsymbol{r}_h \rangle}{E - E_n}}$$

➢ Quark channel contribution. Sum of discrete eigenstates E_n

◆ Energy dependent potential (denominator depends on E)
◆ Non-local potential (numerator depends on $\boldsymbol{r}, \boldsymbol{r}'$ independently)

✓Focus only on the 2nd term which represents the contribution of channel coupling

$X(3872)$

- Construct the model of $X(3872)$
  - ◇ Quark channel : $\bar{c}c$ ◇ Hadron channel : $D^0\,\bar{D}^{*0}$

$$\langle \phi_0 \mid V^t \mid \boldsymbol{r}_h \rangle = g_0 V(\boldsymbol{r}) = g_0 \frac{e^{-\mu r}}{r}$$

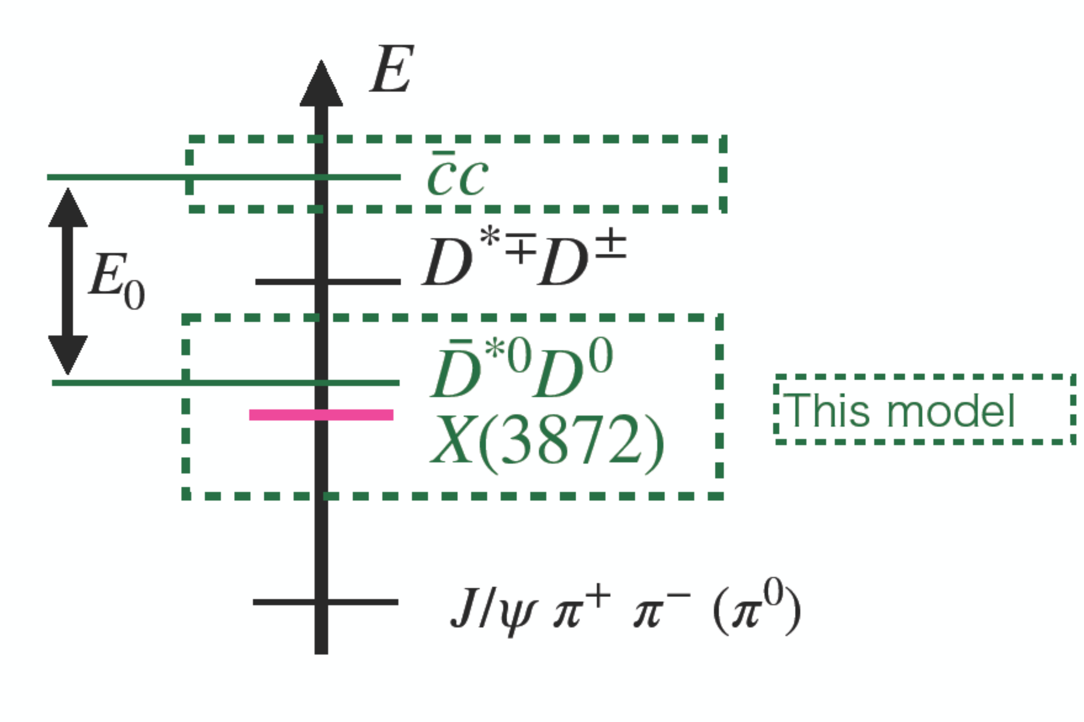
g_0: coupling constant
μ: cut-off

E_0: Energy relative to the threshold of the $D^0\bar{D}^{*0}$ channel

- Effective hadron potential with only $\chi_{C1}(2P)$ contribution (the strongest effect on $D^0\bar{D}^{*0}$)

$$V^h_{\text{eff}}(\boldsymbol{r}, \boldsymbol{r}', E) = \frac{g_0^2}{E - E_0} \frac{e^{-\mu r'}}{r'} \frac{e^{-\mu r}}{r}$$

- Cut-off μ is taken to be mass of π
  - ➢ Lightest exchanging meson
- Energy of $c\bar{c}$: $E_0 = m_{c\bar{c}} - (m_{D^0} + m_{\bar{D}^{*0}})$
- Coupling constant g_0 is determined to reproduce mass of $X(3872)$



$m_{c\bar{c}}$ [S. Godfrey and N. Isgur, Phys. Rev. D, 32, 189 (1985)]
others [PDG Live]

Local approximations

✓Approximation of non-local potential to local one by two different methods

[S.Aoki and K.Yazaki, PTEP 2022, no.3, 033B04 (2022)]

① Formal derivative expansion

- Express non-local potential in terms of derivatives of delta function by Taylor expansion at $\boldsymbol{r} = \boldsymbol{r}'$ directly

② Derivative expansion by HAL QCD method

- Construct the potential from wave function $\psi_{k_0}(r)$ obtained from Schrödinger equation with non-local potentials at momentum k_0

➢Solve for potentials inversely to construct the local potentials

Local approximation for $X(3872)$

✓Converted local potentials from non-local potential $V_{\text{eff}}^{\bar{D}^* D}(\boldsymbol{r}, \boldsymbol{r'}, E)$ in **leading order**

① Formal derivative expansion

$$\boxed{V^{\text{formal}}(r, E) = \frac{4\pi g_0^2}{\mu^2(E - E_0)} \frac{e^{-\mu r}}{r} + \mathcal{O}(\nabla)}$$

② Derivative expansion by HAL QCD method

$$\boxed{V^{\text{HAL}}(r; k_0) = \frac{k_0^2}{2m} + \frac{-k_0^2 \sin\left[k_0 r + \delta(k_0)\right] - \mu^2 \sin\delta(k_0) e^{-\mu r}}{2m\{\sin\left[k_0 r + \delta(k_0)\right] - \sin\delta(k_0) e^{-\mu r}\}}}$$

- $\psi_{k_0}(r)$ and phase shift δ can be solved analytically

- at $E = k_0^2/(2m)$, we can obtain the exact phase shift

Note, $$V^{\text{HAL}}(r; k_0 = 0) = \frac{a_0 \mu^2 e^{-\mu r}}{2m\left(r - a_0 + a_0 e^{-\mu r}\right)} + \mathcal{O}(\nabla^2)$$

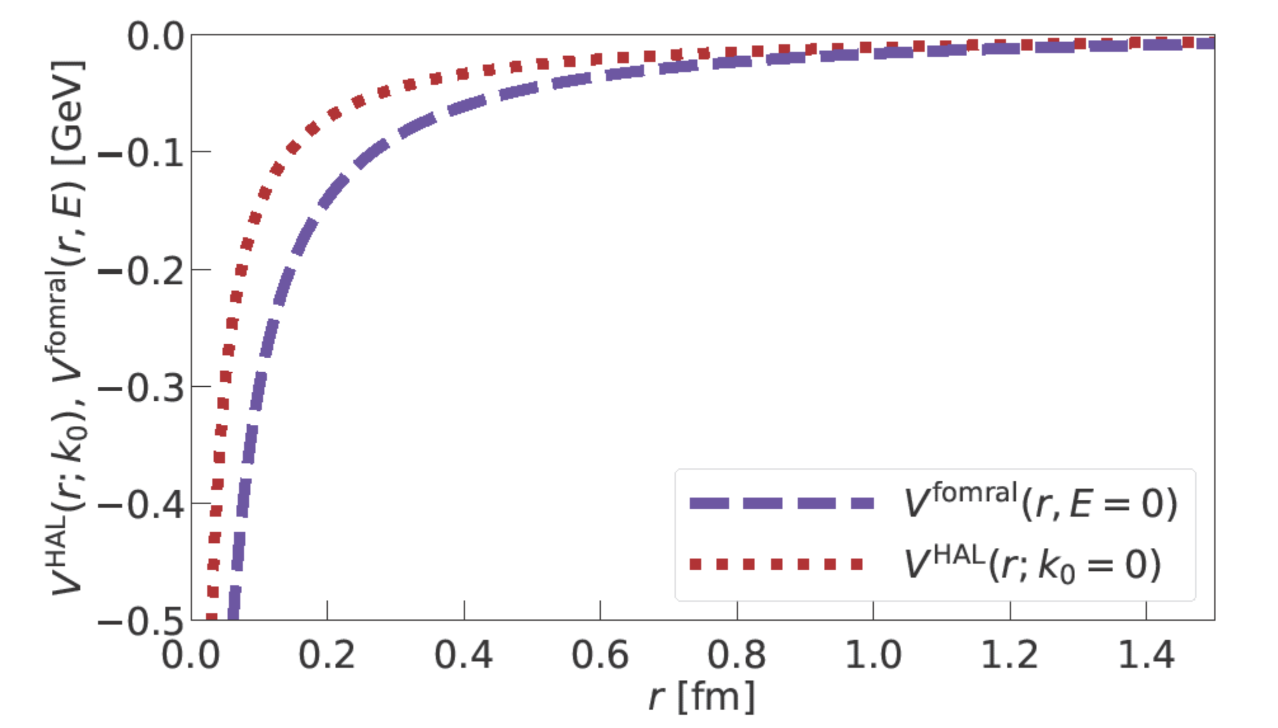
a_0 : scattering length

Result : comparison of V^{HAL} and V^{formal}

- Compare approximated potentials for $X(3872)$
  - V^{HAL} and V^{formal} from the same non-local potential

➢ Both potentials are short-range attraction

➢ Strengths of potential are quantitatively different

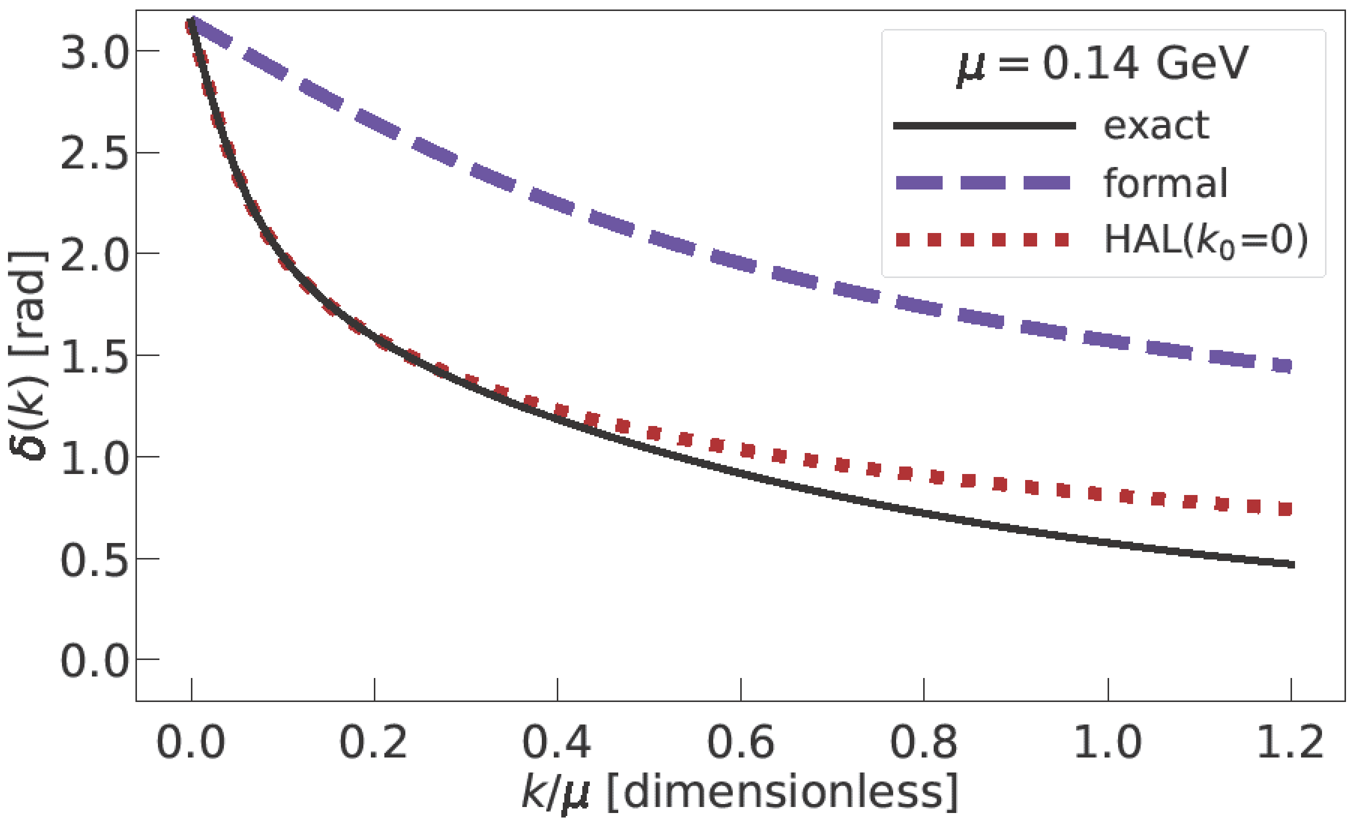


✓How about physical observables from these potentials?

Result : Phase sift $\delta(k)$

●Compare phase shifts $\delta(k)$

- Compare phase shifts $\delta(k)$ from $V^{\text{formal}}(r, E)$ and $V^{\text{HAL}}(r; k_0 = 0)$ with exact $\delta(k)$ from non-local potential



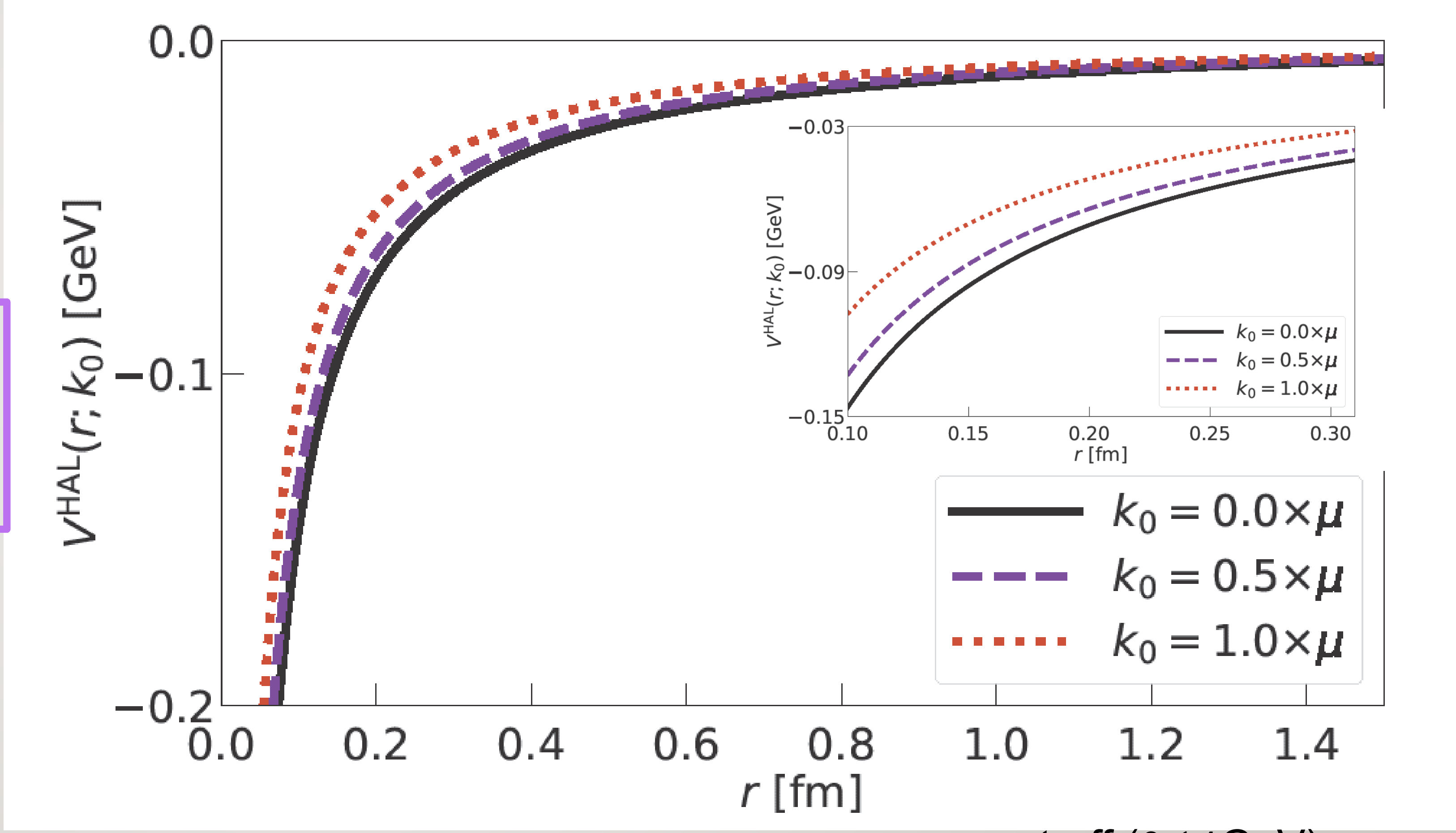
- $\delta(k)$ from HAL QCD method reproduces exact $\delta(k)$, especially for small k

Result : k_0 dependance of V^{HAL} for $X(3872)$

- Local potentials
  by HAL QCD method

$$V_{\rm eff}^{\bar{D}^* D}(\boldsymbol{r}, \boldsymbol{r}', E) \to V^{\rm HAL}(r; k_0)$$

Qualitatively small k_0 dependance



μ : cut-off (0.14GeV)

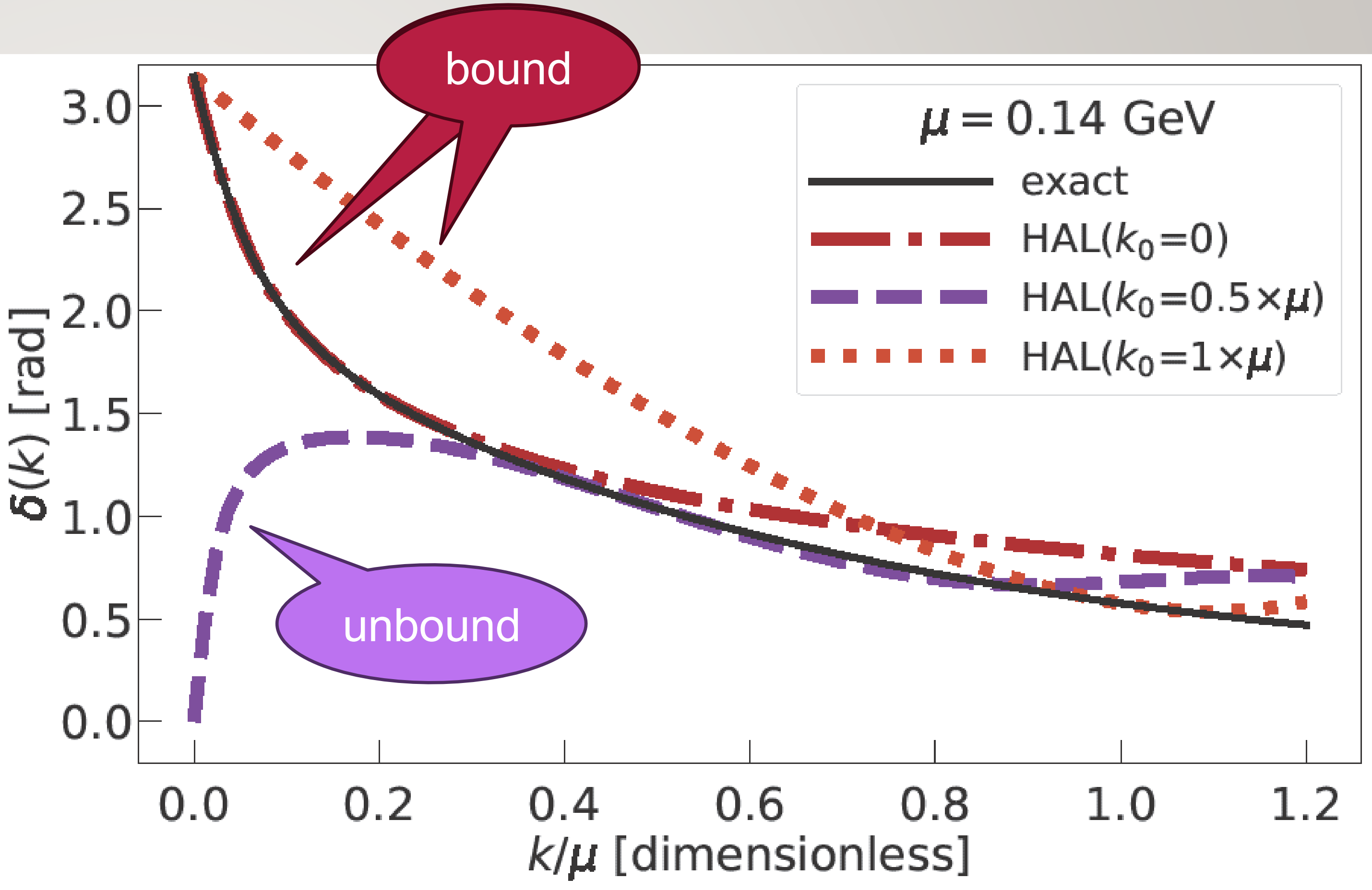
Result : k_0 dependence of $\delta(k)$

- V^{HAL} does not so depend on k_0

- Phase shift $\boldsymbol{\delta}(\boldsymbol{k})$ from $V^{\mathrm{HAL}}(r;k_0)$ qualitatively depends on k_0 strongly

Reason:

- Binding energy of $X(3872)$ is quite small (about 40 keV) so that the phase shift is sensitive to make X(3872) from bound to unbound



Summary

[I. Terashima and T. Hyodo, arXiv:2305.10689 [hep-ph]]

- Channel coupling between quark and hadron d.o.f → ◆ Result : channel coupling
- Channel coupling between $c\bar{c}$ and $D\bar{D}^*$ in $X(3872)$
- Convert non-local to local by 2 methods

  (i) formal derivative expansion, (ii)HAL QCD method

> ➢ Energy dependent
> ➢ Non-local potential

✓ $V^{\rm formal}$ and $V^{\rm HAL}$ are quantitatively different

➢ $V^{\rm HAL}(r;k_0)$ reproduces the exact $\delta(k)$ better than $V^{\rm formal}$

✓ $V^{\rm HAL}(r;k_0)$ has quite small k_0 dependence

➢ $\delta(k)$ from $V^{\rm HAL}(r;k_0)$ is qualitatively depends on k_0 strongly

Future outlook

◆ Append more realistic hadron-hadron interaction

◆ Investigate the influence of the hadron d.o.f on the quark-antiquark effective potentials

Back up

Comparison of a_0

TABLE I. The scattering lengths from the local potentials by the formal derivative expansion (formal) and by the HAL QCD method with $k_0 = 0$ (HAL QCD), in comparison with the exact scattering length from the original nonlocal potential.

	formal	HAL QCD	exact
scattering length [fm]	6.55	24.48	24.48

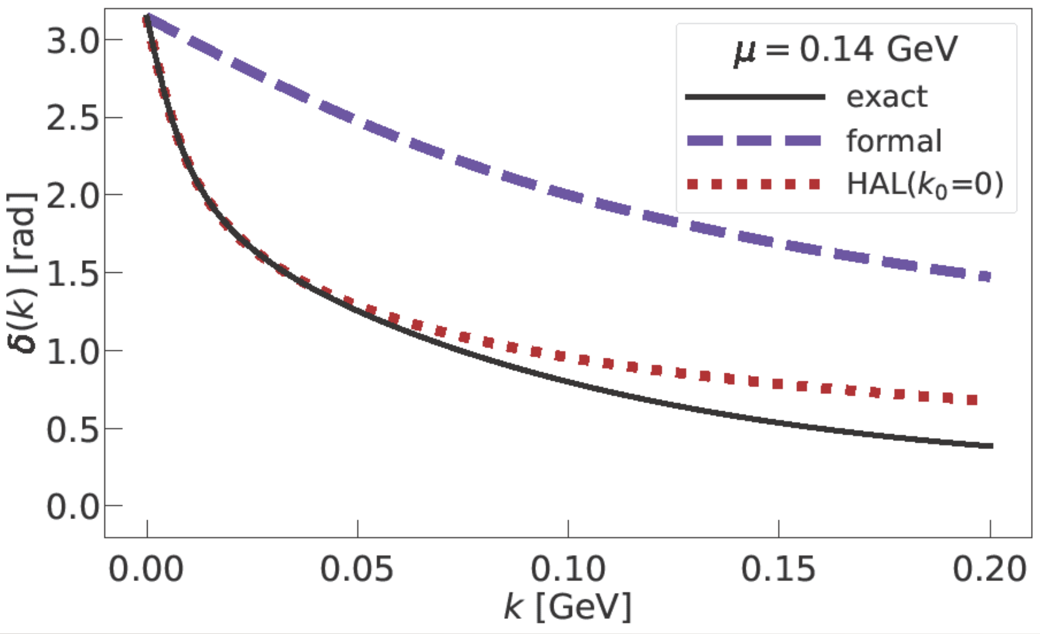
k_0 dependence of a_0

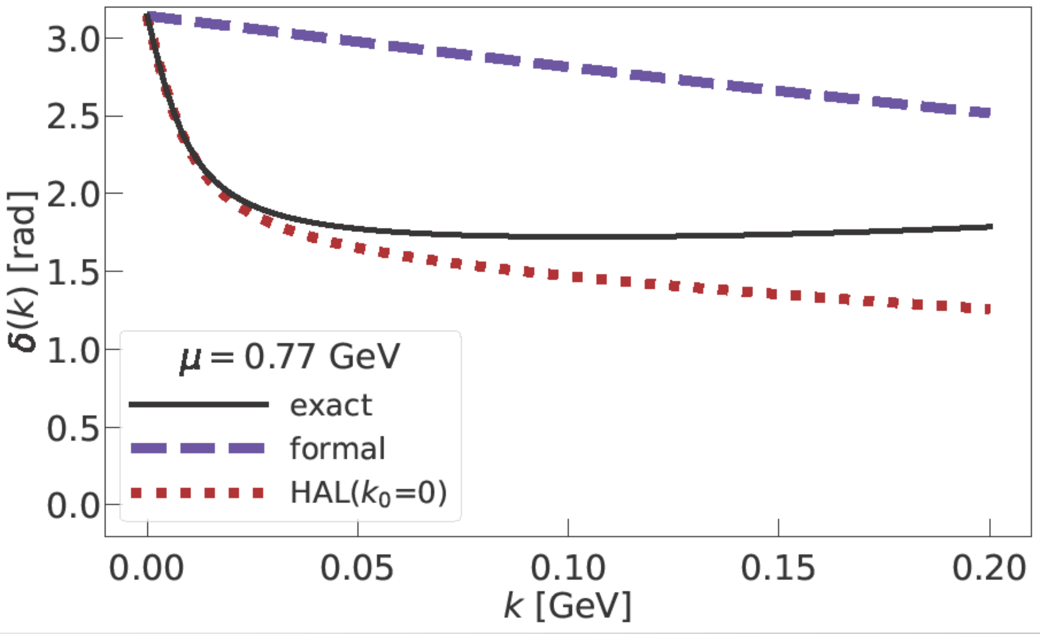
TABLE II. The k_0 dependence of the scattering length a_0 from the potential by the HAL QCD method, with $\mu = m_\pi = 0.14$ GeV and $\mu = m_\rho = 0.77$ GeV.

k_0/μ [dimensionless]	$a_0(\mu = m_\pi)$ [fm]	$a_0(\mu = m_\rho)$ [fm]
0	24.48	22.36
0.1	24.14	8.32
0.2	21.38	2.84
0.3	22.68	1.34
0.4	17.17	0.79
0.5	−63.97	0.71
0.6	9.33	0.01
0.7	5.88	0.23
0.8	−0.78	0.60
0.9	−1.27	−0.13
1	5.21	−0.20

Result : μ dependance of δ

- Change μ but fix the binding energy of $X(3872)$ and $\boxed{k_{\text{pot}} = \sqrt{2mE_{\text{pot}}} = 0}$





- At any μ, V^{HAL} reproduces exact $\delta(k)$ in low energy
- $V^{\text{HAL}}(\boldsymbol{r}, E_{\text{pot}} = 0)$ reproduce exact scattering length a_0 in any μ

Result : effective potential

- Coordinate representation with initial relative coordinate $\boldsymbol{r}$ and final $\boldsymbol{r}'$

$$\langle \boldsymbol{r}'_h \mid V^h_{\text{eff}}(E) \mid \boldsymbol{r}_h \rangle = \langle \boldsymbol{r}'_h \mid V^h \mid \boldsymbol{r}_h \rangle + \boxed{\sum_n \frac{\langle \boldsymbol{r}'_h \mid V^t \mid \phi_n \rangle \langle \phi_n \mid V^t \mid \boldsymbol{r}_h \rangle}{E - E_n}}$$

➢ Quark channel contribution. Sum of discrete eigenstates E_n

$$\langle \boldsymbol{r}'_q \mid V^q_{\text{eff}}(E) \mid \boldsymbol{r}_q \rangle = \langle \boldsymbol{r}'_q \mid V^q \mid \boldsymbol{r}_q \rangle + \boxed{\int d\boldsymbol{p} \frac{\langle \boldsymbol{r}'_q \mid V^t \mid \boldsymbol{p}_{\text{full}} \rangle \langle \boldsymbol{p}_{\text{full}} \mid V^t \mid \boldsymbol{r}_q \rangle}{E - E_{\boldsymbol{p}} + i0^+}}$$

➢ Hadron channel contribution. Integral of continuous eigenstates E_p

◆ Energy dependent potential (denominator depends on E)
◆ Non-local potential (numerator depends on $\boldsymbol{r}, \boldsymbol{r}'$ independently)

I. Terashima and T. Hyodo, EPJ Web Conf. **271**, 10004 (2022)

✓Focus only on the 2nd term which represents the contribution of channel coupling

HAL QCD method in detail

- Schrödinger equation with Yukawa-Separable **non-local potential**

Energy dependent

$$-\frac{1}{2m}\nabla^2 \underline{\psi_{k_{\rm pot}}(\boldsymbol{r})} + \int d^3\boldsymbol{r}' V_{\rm eff}^{\bar{D}^* D}(\boldsymbol{r}, \boldsymbol{r}', E) \underline{\psi_{k_{\rm pot}}(\boldsymbol{r}')} = E_{\rm pot} \underline{\psi_{k_{\rm pot}}(\boldsymbol{r})}$$

未知数は $\psi_{k_{\rm pot}}(\boldsymbol{r})$

$$V_{\rm eff}^{\bar{D}^* D}(\boldsymbol{r}, \boldsymbol{r}', E) = \frac{g_0^2}{E - E_0} \frac{e^{-\mu r}}{r} \frac{e^{-\mu r'}}{r'}$$

Obtain wavefunction $\psi_{k_{\rm pot}}(\boldsymbol{r})$

- $\psi_{k_{\rm pot}}(\boldsymbol{r})$が、**local**ポテンシャルを用いたSchrödinger方程式に従うと仮定

$$-\frac{1}{2m}\nabla^2 \psi_{k_{\rm pot}}(\boldsymbol{r}) + \underline{V_{E_{\rm pot}}^{\rm HAL}(r,)}\psi_{k{\rm pot}}(\boldsymbol{r}) = E_{\rm pot}\psi_{k{\rm pot}}(\boldsymbol{r})$$

未知数は $V_0^{\rm pot}(r, E_{\rm pot})$

- localポテンシャルについて逆解き

$E = E_{\rm pot}$ でlocal Schrödingerを解くと $\psi = \psi_{k_{\rm pot}}$ を厳密に得るので、non-local V位相差を再現

表 6.1　パラメーターの値

物理量	値	出典
$m_{c\bar{c}}$	3.950 GeV	クォーク模型の値 [24]
m_{D^0}	1.86484 GeV	PDG [60]
$m_{D^{0*}}$	2.00685 GeV	PDG [60]
μ	0.14 GeV	PDG [60]
$\hbar c$	0.1973269804 GeV · fm	PDG [60]
$m_{X(3872)}$	3.87165 GeV	PDG [60]

$\chi_{c1}(2P)$

表 6.2　数値計算で得られたパラメーター

物理量	値	計算式
E_0	0.07831 GeV	$m_{c\bar{c}} - m_{D^0} - m_{\bar{D}^{*0}}$
m	0.9666 GeV	$\frac{m_{D^0}+m_{\bar{D}^{*0}}}{m_{D^0}m_{\bar{D}^{*0}}}$
g_0^2	1.999×10^{-5} GeV^3	$\frac{m_{c\bar{c}}-m_{X(3872)}}{I}$
a_0	124.1 fm	$\frac{8\pi m g_0^2/E_0}{\mu(4\pi m g_0^2/E_0+\mu^3)}$

Result : E dependance of V^{formal} for $X(3872)$

- Local potentials
  by formal derivative expansion

$$V_{\text{eff}}^{\bar{D}^* D}(\boldsymbol{r}, \boldsymbol{r'}, E) \to V^{\text{formal}}(r, E)$$

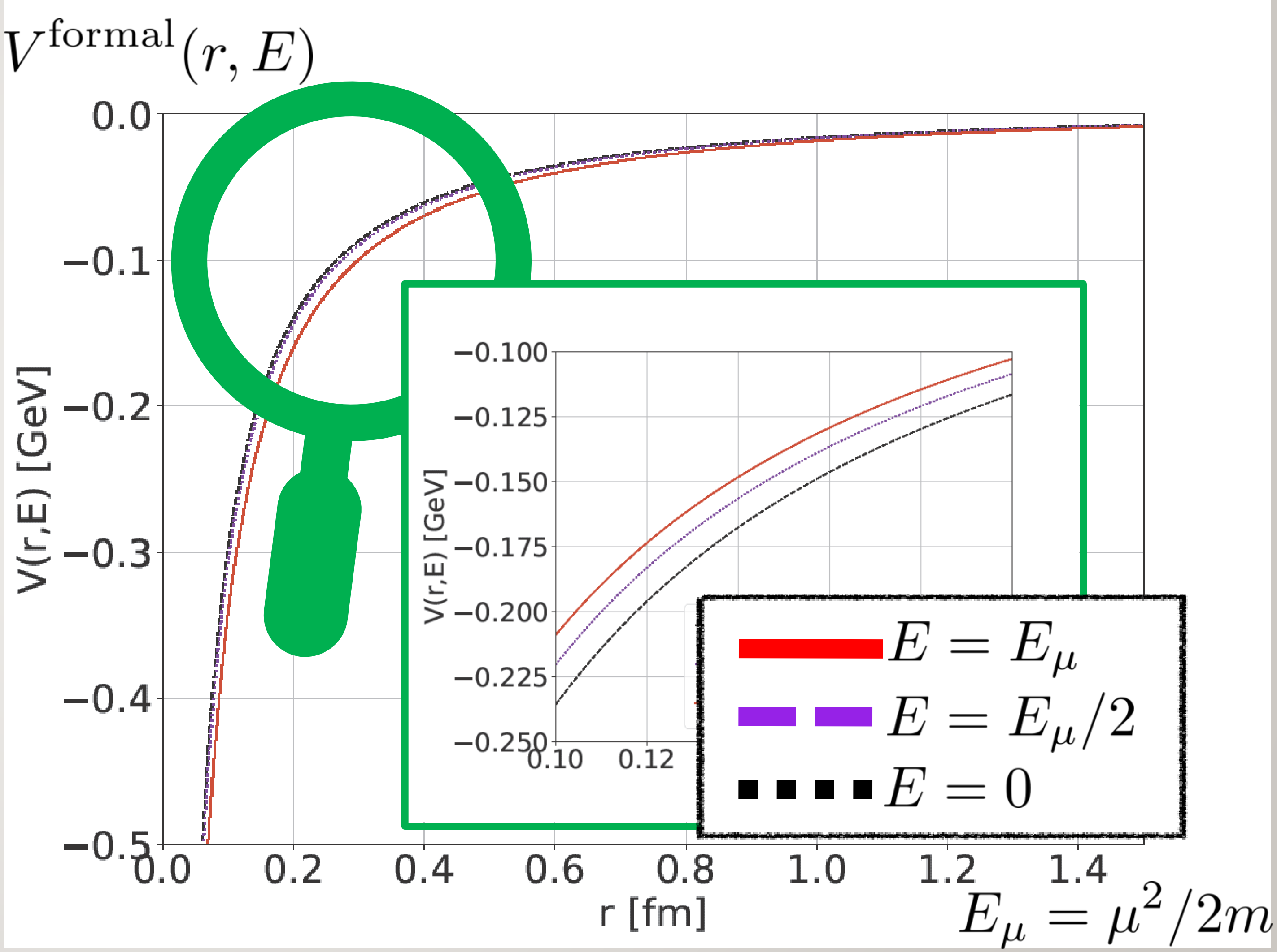
Small E dependance

Reason:

$$V^{\text{formal}}(r, E) = \frac{4\pi g_0^2}{\mu^2 (E - E_0)} \frac{e^{-\mu r}}{r}$$

$$E \ll E_0 = 0.078\text{GeV}$$

$(E_\mu \simeq 0.01 \text{ GeV})$



$E_\mu = \mu^2/2m$

μ : cut-off (0.14GeV)