Precision tests of fundamental physics with $\eta$ and $\eta'$ mesons

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Indiana University/Jefferson Laboratory

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Outline

1. Introduction and Motivation
2. $\eta \rightarrow 3\pi$ and light quark masses
3. $\eta' \rightarrow \eta\pi\pi$ and chiral dynamics
4. Conclusion and Outlook
1. Introduction and Motivation
1.1 Why is it interesting to study $\eta$ and $\eta'$ physics?

- In the study of $\eta$ and $\eta'$ physics, large amount of data have been collected:
  
  $\text{CBall, WASA, KLOE & KLOEII, BESIII, A2@MAMI, CLAS, GlueX}$

  More to come: $\text{JEF, REDTOP}$

- Unique opportunity:
  
  - Test chiral dynamics at low energy
  - Extract fundamental parameters of the Standard Model: ex: light quark masses
  - Study of fundamental symmetries: C, P & T violation
  - Looking for beyond Standard Model Physics
# Rich physics program at \(\eta,\eta'\) factories

## Standard Model highlights
- Theory input for light-by-light scattering for \((g-2)_\mu\)
- Extraction of light quark masses
- QCD scalar dynamics

## Fundamental symmetry tests
- \(\text{P,CP violation}\)
- \(\text{C,CP violation}\)

[[Kobzarev & Okun (1964), Prentki & Veltman (1965), Lee (1965), Lee & Wolfenstein (1965), Bernstein et al (1965)]]

## Dark sectors (MeV—GeV)
- Vector bosons
- Scalars
- Pseudoscalars (ALPs)

(Plus other channels that have not been searched for to date)

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Fundamental symmetry tests

- $P,CP$ violation
- $C,CP$ violation

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**Notes:**

- $\eta$ is a pseudoscalar meson
- $\eta'$ is a pseudoscalar meson
- $\chi$PT: Chiral Perturbation Theory
- ALPs: Axion-like particles
- BSM: Beyond the Standard Model

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Dark sectors (MeV—GeV)
- Vector bosons
- Scalars
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(Plus other channels that have not been searched for to date)

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From S. Tulin

Gan, Kubis, E. P., Tulin’20
2. $\eta \to 3\pi$ and light quark mass extraction

*In collaboration with G. Colangelo, S. Lanz and H. Leutwyler (ITP-Bern)*

*Phys. Rev. Lett. 118 (2017) no.2, 022001*

2.1 Decays of $\eta$

- $\eta$ decay from PDG:

$$M_\eta = 547.862(17) \text{ MeV}$$

### $\eta$ DECAY MODES

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<td>$(72.12 \pm 0.34)%$</td>
<td>S=1.2</td>
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<td>$\Gamma_2$ $2\gamma$</td>
<td>$(39.41 \pm 0.20)%$</td>
<td>S=1.1</td>
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<td>$\Gamma_3$ $3\pi^0$</td>
<td>$(32.68 \pm 0.23)%$</td>
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**Neutral modes**

**Charged modes**

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<td>$\Gamma_8$ charged modes</td>
<td>$(28.10 \pm 0.34)%$</td>
<td>S=1.2</td>
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<td>$\Gamma_9$ $\pi^+ \pi^- \pi^0$</td>
<td>$(22.92 \pm 0.28)%$</td>
<td>S=1.2</td>
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<tr>
<td>$\Gamma_{10}$ $\pi^+ \pi^- \gamma$</td>
<td>$(4.22 \pm 0.08)%$</td>
<td>S=1.1</td>
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2.1 Why is it interesting to study $\eta \rightarrow 3\pi$?

- Decay forbidden by isospin symmetry

$$A = (m_u - m_d) A_1 + \alpha_{em} A_2$$

- $\alpha_{em}$ effects are small

  *Sutherland’66, Bell & Sutherland’68
  *Baur, Kambor, Wyler’96, Ditsche, Kubis, Meissner’09

- Decay rate measures the size of isospin breaking $(m_u - m_d)$ in the SM:

  $$L_{QCD} \rightarrow L_{IB} = -\frac{m_u - m_d}{2} (\bar{u}u - \bar{d}d)$$

  Unique access to $(m_u - m_d)$
2.1 Definitions

- **η decay**: $\eta \rightarrow \pi^+ \pi^- \pi^0$

  \[
  \langle \pi^+ \pi^- \pi^0_{out} | \eta \rangle = i (2\pi)^4 \delta^4 \left( p_\eta - p_{\pi^+} - p_{\pi^-} - p_{\pi^0} \right) A(s,t,u)
  \]

- Mandelstam variables
  
  \[
  s = \left( p_{\pi^+} + p_{\pi^-} \right)^2, \quad t = \left( p_{\pi^-} + p_{\pi^0} \right)^2, \quad u = \left( p_{\pi^0} + p_{\pi^+} \right)^2
  \]

  - only two independent variables

- 3 body decay $\rightarrow$ Dalitz plot

  \[
  |A(s,t,u)|^2 = N \left( 1 + aY + bY^2 + dX^2 + fY^3 + ... \right)
  \]

  Expansion around $X = Y = 0$

  \[
  X = \sqrt{3} \frac{T_+ - T}{Q_c} = \frac{\sqrt{3}}{2M_\eta Q_c} (u - t)
  \]

  \[
  Y = \frac{3T_0}{Q_c} - 1 = \frac{3}{2M_\eta Q_c} \left( \left( M_\eta - M_{\pi^0} \right)^2 - s \right) - 1
  \]

  \[
  Q_c \equiv M_\eta - 2M_{\pi^+} - M_{\pi^0}
  \]
2.2 Quark mass ratio

• In the following, extraction of $Q$ from $\eta \rightarrow \pi^+ \pi^- \pi^0$

\[
\Gamma_{\eta \rightarrow \pi^+ \pi^- \pi^0} = \frac{1}{Q^4} \frac{M_K^4}{M_\pi^4} \left( \frac{M_K^2 - M_\pi^2}{6912}\pi^3 F_\pi^4 \right)^2 M_\eta^3 \int_{s_{\min}}^{s_{\max}} ds \int_{u_{-}(s)}^{u_{+}(s)} du \left| M(s,t,u) \right|^2
\]

Determined from experiment

Determined from:
- Dispersive calculation
- ChPT

Fit to Dalitz distr.

\[
Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}
\]

\[
\hat{m} \equiv \frac{m_d + m_u}{2}
\]

• Aim: Compute $M(s,t,u)$ with the best accuracy
2.3 Computation of the amplitude

- What do we know?
- Compute the amplitude using ChPT:

\[
\Gamma_{\eta \rightarrow 3\pi} = (66 + 94 + ... + ...) \text{eV} = (300 \pm 12) \text{eV}
\]

The Chiral series has convergence problems

LO: Osborn, Wallace’70
NLO: Gasser & Leutwyler’85
NNLO: Bijnens & Ghorbani’07

PDG’16

Anisovich & Leutwyler’96
2.3 Computation of the amplitude

• What do we know?
• The amplitude has an Adler zero: soft pion theorem
  \[ \text{Amplitude has a zero for:} \]
  \[ p_{\pi^+} \rightarrow 0 \quad s = u = 0, \quad t = M_{\eta}^2 \quad M_{\pi} \neq 0 \]
  \[ p_{\pi^-} \rightarrow 0 \quad s = t = 0, \quad u = M_{\eta}^2 \]

\[ s = u = \frac{4}{3} M_{\pi}^2, \quad t = M_{\eta}^2 + \frac{M_{\pi}^2}{3} \]

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SU(2) corrections

Adler’85

Anisovich & Leutwyler’96
2.4 Neutral channel: $\eta \rightarrow \pi^0 \pi^0 \pi^0$

- What do we know?
- We can relate charged and neutral channels

$$A(s,t,u) = A(s,t,u) + A(t,u,s) + A(u,s,t)$$

Correct formalism should be able to reproduce both charged and neutral channels

- Ratio of decay width precisely measured

$$r = \frac{\Gamma(\eta \rightarrow \pi^0 \pi^0 \pi^0)}{\Gamma(\eta \rightarrow \pi^+ \pi^- \pi^0)} = 1.426 \pm 0.026 \quad PDG'19$$
2.4 Neutral Channel: $\eta \rightarrow \pi^0 \pi^0 \pi^0$

- Decay amplitude:

$$\Gamma_{\eta \rightarrow 3\pi} \propto |A|^2 \propto 1 + 2\alpha Z$$

with

$$Z = \frac{2}{3} \sum_{i=1}^{3} \left( \frac{3T_i}{Q_n} - 1 \right)^2$$

$Q_n \equiv M_\eta - 3M_{\pi^0}$

**Important discrepancy** between ChPT and experiment!

- Help of a dispersive treatment?

$$\alpha = -0.0288 \pm 0.0012$$

- $\chi$PT $O(p^4)$ Bijnens & Gasser (2002)
- $\chi$PT $O(p^6)$, Bijnens & Ghorbani (2007)

- Crystal Barrel@LEAR (1998)
- Crystal Ball@BNL (2001)
- SND (2001)
- WASA@CELSIUS (2007)
- WASA@COSY (2008)
- Crystal Ball@MAMI-B (2009)
- Crystal Ball@MAMI-C (2009)
- KLOE (2010)
- BESIII(2015)
- A2-MAMI(2018)
- PDG average

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2.5 Dispersive treatment

- The Chiral series has convergence problems

Large $\pi\pi$ final state interactions \cite{Roiesnel&Truong81}
2.5 Dispersive treatment

- The Chiral series has convergence problems

  Large $\pi\pi$ final state interactions  
  
  [Image of a diagram showing $\eta$ decaying into $\pi^+, \pi^0, \pi^-$]

  $\eta = \pi^+ + \pi^- + \pi^0 + \cdots$

- Dispersive treatment:
  - analyticity, unitarity and crossing symmetry
  - Take into account all the rescattering effects
2.6 Why a new dispersive analysis?

• Several new ingredients:
  – **New inputs** available: extraction $\pi\pi$ phase shifts has improved
    Ananthanarayan et al’01, Colangelo et al’01
    Descotes-Genon et al’01
    Kaminsky et al’01, Garcia-Martin et al’09

  – **New experimental programs**, precise Dalitz plot measurements
    TAPS/CBall-MAMI (Mainz), WASA-Celsius (Uppsala), WASA-Cosy (Juelich)
    CBall-Brookhaven, CLAS, GlueX (JLab), KLOE I-II (Frascati)
    BES III (Beijing)

  – **Many improvements** needed in view of very precise data: inclusion of
    – Electromagnetic effects ($O(e^2m)$) Ditsche, Kubis, Meissner’09
    – Isospin breaking effects Gullstrom, Kupsc, Rusetsky’09, Schneider, Kubis, Ditsche’11
    – Inelasticities Albaladejo & Moussallam’15
2.7 Method

- S-channel partial wave decomposition
  \[ A_\lambda(s, t) = \sum_{J}^\infty (2J + 1)d^J_{\lambda,0}(\theta_s)A_J(s) \]

- One truncates the partial wave expansion: \( \rightarrow \) Isobar approximation

\[ A_\lambda(s, t) = \sum_{J}^{J_{\text{max}}} (2J + 1)d^J_{\lambda,0}(\theta_s)f_J(s) \]
\[ + \sum_{J}^{J_{\text{max}}} (2J + 1)d^J_{\lambda,0}(\theta_t)f_J(t) \]
\[ + \sum_{J}^{J_{\text{max}}} (2J + 1)d^J_{\lambda,0}(\theta_u)f_J(u) \]

3 BWs \((\rho^+, \rho^-, \rho^0)\) + background term

\( \rightarrow \) Improve to include final states interactions

Emilie Passemar
2.7 Method

- S-channel partial wave decomposition

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- Use a Khuri-Treiman approach or dispersive approach

  Restore 3 body unitarity and take into account the final state interactions in a systematic way
2.8 Representation of the amplitude

- **Decomposition** of the amplitude as a function of isospin states

\[
M(s,t,u) = M_0(s) + (s-u)M_1(t) + (s-t)M_1(u) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s)
\]

- \(M_I\) isospin \(I\) rescattering in two particles
- Amplitude in terms of S and P waves exact up to NNLO (\(\mathcal{O}(p^6)\))
- Main two body rescattering corrections inside \(M_I\)

Fuchs, Sazdjian & Stern’93
Anisovich & Leutwyler’96
### 2.8 Representation of the amplitude

- **Decomposition** of the amplitude as a function of isospin states

\[
M(s, t, u) = M_0^0(s) + (s - u)M_1^1(t) + (s - t)M_1^1(u) + M_0^2(t) + M_0^2(u) - \frac{2}{3}M_0^2(s)
\]

- **Unitarity relation**:

\[
\text{disc} \left[ M_\ell^I(s) \right] = \rho(s)t_\ell^*(s) \left( M_\ell^I(s) + \hat{M}_\ell^I(s) \right)
\]

---

**Graphical Representation**

- **Right-hand cut**
- **Left-hand cut**

---

**Roy analysis**

*Colangelo et al.’01*
2.8 Representation of the amplitude

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\[ M(s,t,u) = M_0(s) + (s-u)M_1(t) + (s-t)M_1(u) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s) \]

- **Unitarity relation:**

\[ \text{disc} \left[ M^I_\ell(s) \right] = \rho(s) t^*_\ell(s) \left( M^I_\ell(s) + \hat{M}^I_\ell(s) \right) \]

- **Relation of dispersion to reconstruct the amplitude everywhere:**

\[ M_I(s) = \Omega_I(s) \left( P_I(s) + \frac{s^n}{\pi} \int_0^{\infty} ds' \frac{\sin \delta_I(s') \hat{M}^I_I(s')}{\Omega_I(s')} (s'-s-i\epsilon) \right) \]

\[ \Omega_I(s) = \exp \left( \frac{s}{\pi} \int_0^{\infty} ds' \frac{\delta_I(s')}{s'(s'-s-i\epsilon)} \right) \]

Omnès function

\text{Gasser & Rusetsky'18}

- **P_I(s)** determined from a fit to NLO ChPT + experimental Dalitz plot
2.9 $\eta \to 3\pi$ Dalitz plot

- In the charged channel: experimental data from WASA, KLOE, BESIII

- New data expected from CLAS and GlueX with very different systematics

\[ X = \sqrt{3} \frac{T_+ - T_-}{Q_c} = \frac{\sqrt{3}}{2M_{\eta}Q_c}(u - t) \]

\[ Y = \frac{3T_{0}}{Q_c} - 1 = \frac{3}{2M_{\eta}Q_c} \left( \left( M_{\eta} - M_{\pi^0} \right)^2 - s \right) - 1 \]
2.10 Results: Amplitude for $\eta \rightarrow \pi^+ \pi^- \pi^0$ decays

- The amplitude along the line $s = u$:

![Graph showing the amplitude along the line $s = u$ with various theoretical predictions and data points.]

- **LO of $\chi$PT (current algebra)**
- **NLO of $\chi$PT**
- **Uncertainty estimate for NLO representation**
- **NNLO of $\chi$PT (Bijnens & Ghorbani 2007)**
- **Kampf et al. 2011**
- **Guo et al. 2016**
- **This work**
2.10 Results: Amplitude for $\eta \rightarrow \pi^+ \pi^- \pi^0$ decays

- The amplitude along the line $t = u$: 

![Graph showing the amplitude along the line $t = u$. The graph includes different theoretical predictions such as LO of $\chi$PT (current algebra), NLO of $\chi$PT, NNLO of $\chi$PT (Bijnens & Ghorbani 2007), Kampf et al. 2011, Guo et al. 2016, and this work. The physical region is indicated on the graph.](image-url)
2.11 $Z$ distribution for $\eta \rightarrow \pi^0 \pi^0 \pi^0$ decays

- The amplitude squared in the neutral channel is

The agreement is **excellent** between our prediction and the data!
2.12 Comparison of results for $\alpha$

$$\alpha = -0.0307 \pm 0.0017$$
## 2.13 Quark mass ratio

<table>
<thead>
<tr>
<th>$Q$</th>
<th>Experimental systematics needs to be taken into account</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>$\eta \rightarrow 3\pi$</td>
</tr>
<tr>
<td>21</td>
<td>$\chi PT \mathcal{O}(p^4)$ (Gasser, Leutwyler’85)</td>
</tr>
<tr>
<td>22</td>
<td>$\chi PT \mathcal{O}(p^6)$ (Bijnens, Ghorbani’07)</td>
</tr>
<tr>
<td>23</td>
<td>dispersive (Anisovich et al.’96)</td>
</tr>
<tr>
<td>24</td>
<td>dispersive (Kambor et al.’96)</td>
</tr>
<tr>
<td></td>
<td>dispersive (Kampf et al.’11)</td>
</tr>
<tr>
<td></td>
<td>disp, single-channel (Albaladejo et al.’17)</td>
</tr>
<tr>
<td></td>
<td>disp, coupled-channel (Albaladejo et al.’17)</td>
</tr>
<tr>
<td></td>
<td>dispersive (Guo et al., JPAC’15’17)</td>
</tr>
<tr>
<td></td>
<td>dispersive (Colangelo et al.’18)</td>
</tr>
<tr>
<td></td>
<td>$Q = 22.1 \pm 0.7$</td>
</tr>
</tbody>
</table>

- kaon mass splitting
  - Weinberg’77
  - Kastner, Neufeld’08

- lattice, FLAG’19
  - $N_f = 2$
  - $N_f = 2 + 1$
  - $N_f = 2 + 1 + 1$
2.14 Light quark masses

- Smaller values for $Q$
- Smaller values for $m_s/m_d$ and $m_u/m_d$ than LO ChPT

$$Q = 22.1 \pm 0.7$$
$$m_u/m_d = 0.44 \pm 0.03$$

### Math Formulas

- $Q = 22.1 \pm 0.7$
- $m_u/m_d = 0.44 \pm 0.03$
- Smaller values for $Q$
- Smaller values for $m_s/m_d$ and $m_u/m_d$ than LO ChPT
2.14 Comparison with Lattice

![Graph comparing lattice data with other sources.](image-url)
2.15 Prospects

- Uncertainties in the quark mass ratio

Can be investigated and reduced at future facilities

**Figure 17:** Experimental status of $\eta \to \gamma \gamma$ Decay Width (keV). The five points on the left are the results from collider experiments [319, 328–331], point 6 represents the Cornell Primako measurement [332]. Point 7 is the projected error for the PrimEx-eta measurement with a $\sim 3\%$ total error, arbitrarily plotted to agree with the average value of previous measurements. Figure reprinted from Ref. [89].

...more content...

Emilie Passemard
3. $\eta' \to \eta \pi \pi$ and chiral dynamics

*In collaboration with*
S. Gonzalez-Solis (Indiana University)
*Eur. Phys. J. C78 (2018) no.9, 758*
3.1 Why is it interesting to study $\eta' \rightarrow \eta \pi \pi$?

$M_{\eta'} = 957.78(6) \text{ MeV}$

<table>
<thead>
<tr>
<th>Channel</th>
<th>Expt. branching ratio</th>
<th>Discussion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta' \rightarrow 2\gamma$</td>
<td>$(2.20 \pm 0.08)%$</td>
<td>chiral anomaly</td>
</tr>
<tr>
<td>$\eta' \rightarrow 3\gamma$</td>
<td>$&lt; 1.0 \times 10^{-4}$</td>
<td>$C, CP$ violation</td>
</tr>
<tr>
<td>$\eta' \rightarrow e^+e^\gamma$</td>
<td>$&lt; 9 \times 10^{-4}$</td>
<td>$\chi$PT, dark photon (BSM)</td>
</tr>
<tr>
<td>$\eta' \rightarrow 2\pi^0$</td>
<td>$&lt; 4 \times 10^{-4}$</td>
<td>$P, CP$ violation</td>
</tr>
<tr>
<td>$\eta' \rightarrow \pi^+\pi^-$</td>
<td>$&lt; 1.8 \times 10^{-5}$</td>
<td>$P, CP$ violation</td>
</tr>
<tr>
<td>$\eta' \rightarrow 3\pi^0$</td>
<td>$(2.14 \pm 0.20)%$</td>
<td>$m_u - m_d$</td>
</tr>
<tr>
<td>$\eta' \rightarrow \pi^+\pi^-\pi^0$</td>
<td>$(3.8 \pm 0.4) \times 10^{-3}$</td>
<td>$m_u - m_d$, $CP$ violation</td>
</tr>
<tr>
<td>$\eta' \rightarrow \eta\pi^+\pi^-$</td>
<td>$(42.6 \pm 0.7)%$</td>
<td>$R\chi$PT, anomaly, $\eta - \eta'$ mixing</td>
</tr>
<tr>
<td>$\eta' \rightarrow \eta\pi^0\pi^0$</td>
<td>$(22.8 \pm 0.8)%$</td>
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</tr>
<tr>
<td>$\eta' \rightarrow \pi^0e^+e^-$</td>
<td>$&lt; 1.4 \times 10^{-3}$</td>
<td>$C$ violation</td>
</tr>
<tr>
<td>$\eta' \rightarrow \pi^+\pi^-e^+e^-$</td>
<td>$(2.4^{+1.3}_{-1.0}) \times 10^{-3}$</td>
<td>$P, CP$ violation</td>
</tr>
<tr>
<td>$\eta' \rightarrow \pi^0\gamma\gamma$</td>
<td>$&lt; 8 \times 10^{-4}$</td>
<td>$\chi$PT, leptophobic $B$ boson (BSM)</td>
</tr>
<tr>
<td>$\eta' \rightarrow \eta e^+e^-$</td>
<td>$&lt; 2.4 \times 10^{-3}$</td>
<td>$C$ violation</td>
</tr>
</tbody>
</table>

PDG'19
Gan, Kubis, E. P., Tulin'20

Emilie Passemar
## 3.1 Why is it interesting to study $\eta' \rightarrow \eta\pi\pi$?

<table>
<thead>
<tr>
<th>$\eta' \rightarrow$</th>
<th>Branching ratio</th>
<th>Discussion</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
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<td>$C$ violation</td>
</tr>
</tbody>
</table>
3.1 Why is it interesting to study $\eta' \rightarrow \eta\pi\pi$?

- Main decay channel of the $\eta'$:
  \[
  \text{BR}(\eta' \rightarrow \eta\pi^0\pi^0) = 22.8(8)\% \quad \text{and} \quad \text{BR}(\eta' \rightarrow \eta\pi^+\pi^-) = 42.6(7)\%.
  \]

- Precise measurements became available: recent results on
  - neutral channel by $A2$ collaboration: $1.2 \times 10^5$ events
  - neutral and charged channel by $BESIII$ collaboration: $351\,016$ events

\[
|A(s,t,u)|^2 = N \left( 1 + aY + bY^2 + dX^2 + fY^3 + \ldots \right)
\]

\[
s = \left( p_{\eta'} - p_\eta \right)^2, \quad t = \left( p_{\eta'} - p_{\pi^+} \right)^2, \quad u = \left( p_{\eta'} - p_{\pi^-} \right)^2
\]

Expansion around $X=Y=0$

\[
X = \sqrt{3} \frac{T_\eta - T_+}{Q_{\eta'}} = \sqrt{3} \frac{2M_\eta Q_{\eta'}}{2M_\eta Q_{\eta'}} (t - u)
\]

\[
Y = \frac{M_\eta + 2M_\pi}{M_\pi} \frac{T_\eta}{Q_{\eta'}} - 1 = \frac{M_\eta + 2M_\pi}{M_\pi} \left( \frac{M_\eta - M_\eta}{2M_\eta Q_{\eta'}} \right)^2 - 1
\]

$Q_{\eta'} \equiv M_\eta - M_\eta - 2M_\pi$
3.1 Why is it interesting to study $\eta' \rightarrow \eta\pi\pi$?

- Main decay channel of the $\eta'$:
  \[
  \text{BR}(\eta' \rightarrow \eta\pi^0\pi^0) = 22.8(8)\%
  \quad \text{and} \quad
  \text{BR}(\eta' \rightarrow \eta\pi^+\pi^-) = 42.6(7)\%.
  \]

- Precise measurements became available: recent results on
  - neutral channel by A2 collaboration: $1.2 \times 10^5$ events
  - Neutral and charged channel by BESIII collaboration: 351 016 events

\[
|A(s,t,u)|^2 = N \left(1 + aY + bY^2 + dX^2 + fY^3 + \ldots\right)
\]

- Studying this decay allows
  - to test any of the extensions of ChPT e.g. resonance chiral theory, Large-$N_C$ U(3) ChPT etc
  - to study the effects of the $\pi\pi\pi$ and $\pi\eta$ final-state interactions
### 3.2 Theoretical Framework

- **U(3) ChPT with resonances at one-loop**

\[
\begin{pmatrix}
\eta \\
\eta'
\end{pmatrix} = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
\eta_8 \\
\eta_1
\end{pmatrix}
\]

![Diagram showing the structure of the decay amplitude](image-url)
3.2 Theoretical Framework

- U(3) ChPT with resonances at one-loop

\[
\begin{pmatrix}
\eta \\
\eta'
\end{pmatrix} = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}\begin{pmatrix}
\eta_8 \\
\eta_1
\end{pmatrix}
\]

Final-state interaction through the N/D unitarization method

\[
\frac{d^3 \Gamma_{\eta' \rightarrow \eta \pi^0 \pi^0}}{d m_{\eta' \pi^0}} \times 10^4
\]

- Leading Order
- Resonances
- + Resonances
- + Loops

Emilie Passemard
3.2 Theoretical Framework

- Unitarity relations

\[ \text{Im} \mathcal{M}_{\eta' \to \eta \pi \pi} = \frac{1}{2} \sum_n (2\pi)^4 \delta^4 (p_\eta + p_1 + p_2 - p_n) \mathcal{T}_{n \to \eta \pi \pi}^* \mathcal{M}_{\eta' \to n} \]

- A dispersive analysis also exists by Isken et al.'17 but here we include D waves as well as kaon loops
3.3 Results

\[ \int |M(s, t)|^2 \, dm_{\eta\pi^0} \]

This work: \( \pi\pi \) and \( \pi\eta \) FSI

This work: ChPT at one-loop+resonances

Cusp at 0.28 GeV

\( 4 m_{\pi^+}^2 \)
3.3 Results

ChPT

\[ a[Y] = -0.095(6) \]
\[ b[Y^2] = 0.005(1) \]
\[ d[X^2] = -0.037(5) \]

Dalitz slope parameters

Final-state interactions

\[ a[Y] = -0.073(7)(5) \]
\[ b[Y^2] = -0.052(1)(2) \]
\[ d[X^2] = -0.052(8)(5) \]

\[ |A(s,t,u)|^2 = N \left( 1 + aY + bY^2 + dX^2 + fY^3 + ... \right) \]
3.3 Results

\[
|A(s,t,u)|^2 = N\left(1 + aY + bY^2 + dX^2 + fY^3 + \ldots\right)
\]
### 3.4 Role of the D-wave $\pi\pi$ FSI

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Analysis I</th>
<th>Fit 1 (with $D$-wave)</th>
<th>Fit 1 (w/o $D$-wave)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_S$</td>
<td></td>
<td>1017(68)(24)</td>
<td>996(66)(25)</td>
</tr>
<tr>
<td>$c_d$</td>
<td></td>
<td>30.4(4.8)(9)</td>
<td>23.3(3.5)(1.5)</td>
</tr>
<tr>
<td>$c_m$</td>
<td>= $c_d$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tilde{c}_d$</td>
<td></td>
<td>17.6(2.8)(5)</td>
<td>13.5(2.0)(9)</td>
</tr>
<tr>
<td>$\tilde{c}_m$</td>
<td>= $\tilde{c}_d$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_{\pi\pi}$</td>
<td>0.76(61)(6)</td>
<td></td>
<td>2.01(1.61)(71)</td>
</tr>
<tr>
<td>$\chi^2_{dof}$</td>
<td>1.12</td>
<td></td>
<td>1.24</td>
</tr>
<tr>
<td>$a[Y]$</td>
<td>$-0.074(7)(8)$</td>
<td></td>
<td>$-0.091(9)(4)$</td>
</tr>
<tr>
<td>$b[Y^2]$</td>
<td>$-0.049(1)(2)$</td>
<td></td>
<td>$-0.013(1)(5)$</td>
</tr>
<tr>
<td>$c[X]$</td>
<td>0</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>$d[X^2]$</td>
<td>$-0.047(8)(4)$</td>
<td></td>
<td>$-0.031(6)(3)$</td>
</tr>
<tr>
<td>$\kappa_{03}[Y^3]$</td>
<td>0.001</td>
<td></td>
<td>0.001</td>
</tr>
<tr>
<td>$\kappa_{21}[YX^2]$</td>
<td>$-0.004$</td>
<td></td>
<td>$-0.001$</td>
</tr>
<tr>
<td>$\kappa_{22}[Y^2X^2]$</td>
<td>0.001</td>
<td></td>
<td>0.0004</td>
</tr>
</tbody>
</table>
3.5 Prospects

- Comparison to BESIII data

- Simultaneous fit by experimental collaborations to the neutral and charged channels etc
4. Conclusion and Outlook
4.1 Conclusion

- $\eta$ and $\eta'$ allows to study the fundamental properties of QCD:
  - Extraction of fundamental parameters of the SM, e.g. light quark masses
  - Study of chiral dynamics

- To studies $\eta$ and $\eta'$ with the best precision: Development of amplitude analysis techniques consistent with analyticity, unitarity, crossing symmetry, dispersion relations allow to take into account all rescattering effects being as model independent as possible combined with ChPT. Provide parametrization for experimental studies

- In this talk, illustration with $\eta \rightarrow 3\pi$ and extraction of the light quark masses and $\eta' \rightarrow \eta\pi\pi$

- Other illustrations in the talk of e.g. B. Kubis
4.2 Outlook

- Apply dispersion relations + (R)ChPT to other modes in the light meson sector
  - $\omega/\varphi \rightarrow 3\pi, \pi\gamma$: Niecknig, Kubis, Schneider’12, Danilkin et al. JPAC’15,’16, Albaladejo et al’’20
  - $\varphi \rightarrow \eta\pi\gamma$: Moussallam, Shekhovtsova in progress
  - $J/\psi \rightarrow \gamma\pi\pi\pi$ and $J/\psi \rightarrow \gamma\K K$ Rodas, Pilloni et al., JPAC in progress
  - $\eta' \rightarrow 3\pi$: Isken, Kubis and Stoffer in progress
  - $e^+e^- \rightarrow \psi(2S)\pi^+\pi^-\gamma, e^+e^- \rightarrow J/\psi\pi^+\pi^-\gamma, e^+e^- \rightarrow h_c\pi^+\pi^-$ Danilkin, Molnar, Vanderhaeghen’19,’20
  - etc…

See talks by B. Kubis, D. Molnar, A. Pilloni,… at this conference
5. Back-up
Experimental Facilities and Role of JLab 12

M. J. Amaryan et al.
CLAS Analysis Proposal, (2014)

<table>
<thead>
<tr>
<th></th>
<th>$e^+ e^- \gamma$</th>
<th>$\pi^+ \pi^- \gamma$</th>
<th>$\pi^+ \pi^- \pi^0$, $\pi^+ \pi^- e^+ e^-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi$</td>
<td>$e^+ e^- \gamma$</td>
<td>$\pi^+ \pi^- \gamma$</td>
<td>$\pi^+ \pi^- \pi^0$, $\pi^+ \pi^- e^+ e^-$</td>
</tr>
<tr>
<td>$\eta$</td>
<td>$e^+ e^- \gamma$</td>
<td>$\pi^+ \pi^- \gamma$</td>
<td>$\pi^+ \pi^- \pi^0$, $\pi^+ \pi^- \eta$, $\pi^+ \pi^- e^+ e^-$</td>
</tr>
<tr>
<td>$\eta'$</td>
<td>$e^+ e^- \gamma$</td>
<td>$\pi^+ \pi^- \gamma$</td>
<td>$\pi^+ \pi^- \pi^0$, $\pi^+ \pi^- \eta$, $\pi^+ \pi^- e^+ e^-$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$\pi^+ \pi^- \gamma$</td>
<td>$\pi^+ \pi^- \gamma$</td>
<td>$\pi^+ \pi^- \pi^0$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>$e^+ e^- \pi^0$</td>
<td>$\pi^+ \pi^- \gamma$</td>
<td>$\pi^+ \pi^- \pi^0$</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>$\pi^+ \pi^- \pi^0$</td>
<td>$\pi^+ \pi^- \pi^0$</td>
<td>$\pi^+ \pi^- \eta$</td>
</tr>
</tbody>
</table>
2.3 Computation of the amplitude

- What do we know?

- Compute the amplitude using ChPT: the effective theory that describe dynamics of the Goldstone bosons (kaons, pions, eta) at low energy

- Goldstone bosons interact weakly at low energy and \( m_u, m_d \ll m_s < \Lambda_{QCD} \)

Expansion organized in external momenta and quark masses

\[
L_{\text{eff}} = \sum_{d \geq 2} L_d, \quad L_d = \mathcal{O}(p^d), \quad p \equiv \{q, m_q\}
\]

**Weinberg’s power counting rule**

\[ p \ll \Lambda_H = 4\pi F_\pi \sim 1 \text{ GeV} \]
2.5 Iterative Procedure

- Solution *linear* in the *subtraction constants*

\[ M(s,t,u) = \alpha_0 M_{\alpha_0}(s,t,u) + \beta_0 M_{\beta_0}(s,t,u) + \ldots \]

\[ \text{makes the fit much easier} \]

Adapted from P. Stoffer’15

\[ \pi\pi \text{ elastic phase shifts } \delta_l^I \]

- fix one subtraction constant to 1, all others to 0
- compute \( \hat{M}_i \) with angular integrals
- compute \( M_i \) with dispersive integrals
- compute Omnès functions \( \Omega^I_l \)
- determination of subtraction constants: fit to data + chiral constraints

Adapted from P. Stoffer’15

**Anisovich & Leutwyler’96**
2.6 Subtraction constants

- Extension of the numbers of parameters compared to *Anisovich & Leutwyler’96*

\[ P_0(s) = \alpha_0 + \beta_0 s + \gamma_0 s^2 + \delta_0 s^3 \]
\[ P_1(s) = \alpha_1 + \beta_1 s + \gamma_1 s^2 \]
\[ P_2(s) = \alpha_2 + \beta_2 s + \gamma_2 s^2 \]

- In the work of *Anisovich & Leutwyler’96* matching to one loop ChPT
  Use of the SU(2) x SU(2) chiral theorem
  The amplitude has an *Adler zero* along the line s=u

- Now data on the Dalitz plot exist from KLOE, WASA, MAMI and BES III
  Use the data to directly fit the subtraction constants

- However normalization to be fixed to ChPT!
2.7 Subtraction constants

- The subtraction constants are

\[ P_0(s) = \alpha_0 + \beta_0 s + \gamma_0 s^2 + \delta_0 s^3 \]
\[ P_1(s) = \alpha_1 + \beta_1 s + \gamma_1 s^2 \]
\[ P_2(s) = \alpha_2 + \beta_2 s + \gamma_2 s^2 + \delta_0 s^3 \]

Only 6 coefficients are of physical relevance.

- They are determined from combining ChPT with a fit to KLOE Dalitz plot.

- Taylor expand the dispersive \( M_i \)

Subtraction constants \( \leftrightarrow \) Taylor coefficients

\[ M_0(s) = A_0 + B_0 s + C_0 s^2 + D_0 s^3 + \ldots \]
\[ M_1(s) = A_1 + B_1 s + C_1 s^2 + \ldots \]
\[ M_2(s) = A_2 + B_2 s + C_2 s^2 + D_2 s^3 + \]

- Gauge freedom in the decomposition of \( M(s,t,u) \)
2.7 Subtraction constants

- Build some gauge independent combinations of Taylor coefficients

\[
H_0 = A_0 + \frac{4}{3} A_2 + s_0 \left( B_0 + \frac{4}{3} B_2 \right)
\]
\[
H_1 = A_1 + \frac{1}{9} (3 B_0 - 5 B_2) - 3 C_2 s_0
\]
\[
H_2 = C_0 + \frac{4}{3} C_2, \quad H_3 = B_1 + C_2
\]
\[
H_4 = D_0 + \frac{4}{3} D_2, \quad H_5 = C_1 - 3 D_2
\]

\[
H_0^{\text{ChPT}} = 1 + 0.176 + O(p^4)
\]
\[
h_1^{\text{ChPT}} = \frac{1}{\Delta_{\eta\pi}} \left( 1 - 0.21 + O(p^4) \right)
\]
\[
h_2^{\text{ChPT}} = \frac{1}{\Delta^2_{\eta\pi}} \left( 4.9 + O(p^4) \right)
\]
\[
h_3^{\text{ChPT}} = \frac{1}{\Delta^2_{\eta\pi}} \left( 1.3 + O(p^4) \right)
\]

\[
\chi^2_{\text{theo}} = \sum_{i=1}^{3} \left( \frac{h_i - h_i^{\text{ChPT}}}{\sigma h_i^{\text{ChPT}}} \right)^2
\]

\[
\sigma h_i^{\text{ChPT}} = 0.3 \left| h_i^{\text{NLO}} - h_i^{\text{LO}} \right|
\]
Isospin breaking corrections

- Dispersive calculations in the isospin limit to fit to data one has to include isospin breaking corrections

\[ M_{c/n}(s,t,u) = M_{\text{disp}}(s,t,u) \frac{M_{\text{DKM}}(s,t,u)}{M_{\text{GL}}(s,t,u)} \]

with \( M_{\text{DKM}} \): amplitude at one loop with \( \mathcal{O}(e^2m) \) effects

\( \text{Ditsche, Kubis, Meissner'09} \)

\( M_{\text{GL}} \): amplitude at one loop in the isospin limit

\( \text{Gasser & Leutwyler’85} \)

Kinematic map:
- isospin symmetric boundaries
- physical boundaries

\[ M_{\text{GL}} \rightarrow \tilde{M}_{\text{GL}} \]

\[ Q_n \equiv M_\eta - 3M_{\pi^0} \]

Emilie Passemar
2.15 Prospects

<table>
<thead>
<tr>
<th>Exp.</th>
<th>$3\pi^0$ Events ($10^6$)</th>
<th>$\pi^+ \pi^- \pi^0$ Events ($10^6$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total world data (include prel. WASA and prel. KLOE)</td>
<td>6.5</td>
<td>6.0</td>
</tr>
<tr>
<td>GlueX+PrimEx-(\eta) +JEF</td>
<td>20</td>
<td>19.6</td>
</tr>
</tbody>
</table>

- Existing data from the low energy facilities are sensitive to the detection threshold effects.
- JEF at high energy has uniform detection efficiency over Dalitz phase space.
- JEF will offer large statistics and different systematics.