Bound states in the three-body scattering formalisms

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Three-body scattering amplitudes?

- Exotic resonances decay to three-particle final states
  - $X(3872)$, $N(1440)$, $a_1(1420)$, ...

- Example: interpretations of $X(3872)$
  - molecule
  - charmonium–molecule hybrid
  - diquark–antidiquark
  - kinematical effect

- Goals
  - Phenomenology
  - Lattice QCD
Three-particles interactions from QCD

Finite volume spectrum through the Quantization Condition gives a three-body K-matrix,

3→3 amplitudes obtained from the K-matrix + knowledge of two-body subprocesses, through the set of integral equations,

Final amplitudes analytically continued to the unphysical Riemann sheets,
Formalisms under construction

- Effective Field Theory framework
  - generic EFT,
  - summation of 2PI, 3PI diagrams,

  Hansen, Sharpe, Phys. Rev. D 90, 116003 (2014),
  Hansen, Sharpe, Phys. Rev. D 92, 114509 (2015),
  Hansen, Sharpe, Phys. Rev. D 95, 034501 (2017),
  and many more...

- Unitarity-based framework
  - parametrization based on the S matrix unitarity,

  and many more...

First results — three pions at I=3


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They are equivalent, both in their IV and FV versions:

The B-matrix approach

- Starting with the S matrix unitarity
- Physical degrees of freedom (domain of integration)
- Simple parametrization with clear interpretation

Three-body amplitude

\[ A_{\ell m_\ell; \ell' m_{\ell'}} (\sigma', s, \sigma) \]
- pair–spectator,
- partial waves,
- symmetrization,
Bound–state–particle scattering

- Model study: formation of the three–body bound states
- Properties of the integral equations and analytic continuation
- Simple model: two–particle bound state + particle, S wave
  - Analytic properties of the B–matrix formalism
    
    \[ \mathcal{F}^{-1} \sim \frac{1}{a} + i\rho \]

    \[ \lim_{\sigma,\sigma'\to\sigma_b} A_3 = \frac{g}{\sigma' - \sigma_b} A_2 \frac{g}{\sigma - \sigma_b} \]

Phys.Rev.D 103 (2021) 1, 014009 (with A. Szczepaniak)

- Numerical solution of the three–body EFT equations
  
Generalization of the B-matrix equations

- One can not use the LSZ reduction — bound-state energies outside of physical region
- Multi-hadron scattering requires generalization to all channels

\[
A_{22} = B_{22} + \int_{k} B_{22} i \rho_2 A_{22} + \int_{q} B_{23,q} A_{32,q},
\]
\[
A_{23,p} = B_{23,p} F_{p} + \int_{k} B_{22} i \rho_2 A_{23,p} + \int_{q} B_{23,q} A_{33,qp},
\]
\[
A_{32,p'} = F_{p'} B_{32,p'} + \int_{k} F_{p'} B_{32,p'} i \rho_2 A_{32,p} + \int_{q} F_{p'} B_{33,qp} A_{32,q},
\]
\[
A_{33,p'^p} = F_{p'} B_{33,p'^p} F_{p} + \int_{k} F_{p'} B_{32,p'} i \rho_2 A_{23,p} + \int_{q} F_{p'} B_{33,qp} A_{33,qp}.
\]

- Satisfies unitarity above the three-particle threshold,

**Approximation:** all multi-particle interactions are constant and real (couplings $g_{ij}$),

\[
a_{33}(s) = \frac{g_{33}}{1 - g_{22}i \rho_2 - g_{33} \mathcal{I}(s)},
\]
\[
a_{22}(s) = \frac{g_{22}}{1 - g_{22}i \rho_2 - g_{33} \mathcal{I}(s)}.
\]
Analyticity of the B-matrix equations

- Solutions do not satisfy unitarity below the three-body threshold
- Spurious singularities start arbitrarily close to the two-body threshold

Kernel suffering from non-physical left-hand cuts:

\[ I(s) = \int_{\sigma_{\text{min}}}^{(\sqrt{s}-m)^2} \frac{d\sigma_q}{2\pi} \tau(s, \sigma_q) F(\sigma_q) \]

Dispersion procedure ensures analyticity

Kernel free from those problems:

\[ I_d(s) = \frac{(s-s_s)^2}{\pi} \int_{(3m)^2}^{\infty} ds' \frac{\text{Im} I(s')}{(s'-s-i\epsilon)(s'-s_s-i\epsilon)^2} \]

- left-hand cut
- two-particle threshold
- three-particle threshold

Bound states in the three-body scattering formalisms
Spurious singularities pushed to non-physical Riemann sheets, can study physics,

General dispersion procedure for the three-body unitarity formalism is needed,
Solving the EFT three-body ladder equation

- Ladder approximation, $B = G + (R=0)$

\[ \begin{align*}
&\quad = \quad \begin{array}{c} \text{\includegraphics[width=0.3\textwidth]{ladder_diagram.png}} \end{array} \\
&\quad = \quad \begin{array}{c} \text{\includegraphics[width=0.3\textwidth]{ladder_diagram.png}} \end{array} + \begin{array}{c} \text{\includegraphics[width=0.3\textwidth]{ladder_diagram.png}} \end{array} + \begin{array}{c} \text{\includegraphics[width=0.3\textwidth]{ladder_diagram.png}} \end{array} + \cdots
\end{align*} \]

- Calculation above the bound-state–particle threshold,

  - weakly interacting system in the $\pi^+\pi^+$ and $\pi^+\pi^+\pi^+$

    Hansen et al., Phys. Rev. Lett. 126 (2021), 012001

  - decay $a_1(1260) \rightarrow \rho^0\pi^- \rightarrow \pi^-\pi^+\pi^-$

**Numerical procedure**

- Discretization of the integral equation → **N linear equations** (Matrix equation)
- Regulation of the bound–state pole via $\epsilon$–prescription

$$A_2(s) = \lim_{\epsilon \to 0^+} \lim_{N \to \infty} A_2(s; N, \epsilon)$$

**Systematics:**

- **Unitarity test:** $\text{Im} A_2(s) = \rho_2(s) |A_2(s)|^2$  →  $\Delta \rho_2 = 100 \times \left| \frac{\text{Im} A_2^{-1}(s; N, \epsilon) + \rho_2(s)}{\rho_2(s)} \right|$

- **Convergence test:** $\Delta_N A_2 = 2 \times \left| \frac{A_2(s; N + 1, \epsilon) - A_2(s; N, \epsilon)}{A_2(s; N + 1, \epsilon) + A_2(s; N, \epsilon)} \right|$

**Methods:**

- "Brute force"
- Explicit pole removal
- Spline–based quadratures

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Extrapolation in $N$

- Expansion in $1/N$,
- Epsilon regulator fixed to $\epsilon \propto \eta/N$

![Graphs showing extrapolation in $N$](image)
Example results, $M^2 = 3m^2$

$\Delta \rho_2 \left( \frac{q}{m} \right) \cot \delta$

$\frac{q}{m}$ vs $\cot \delta$

$\Delta \rho_2$ vs $(E/m)^2$
Example result, three-body bound state

This work, $\eta = 15$
Romero-López et al. 
NREFT $\Lambda = 0.75m$

Roman-Lopez et al., JHEP 10 (2019) 007
Conclusions

- First lattice calculations of the 3-hadron systems
- Systematic procedure for solving the integral equations

Future work

- Continuing below the two-body threshold and to complex energies
- Formulation of the B-matrix satisfying analyticity
- Generalization to arbitrary spins
- Controlling the scheme/cutoff dependence
BACKUP SLIDES
We are not limited below the three-body threshold,