Bound states in the three-body scattering formalisms

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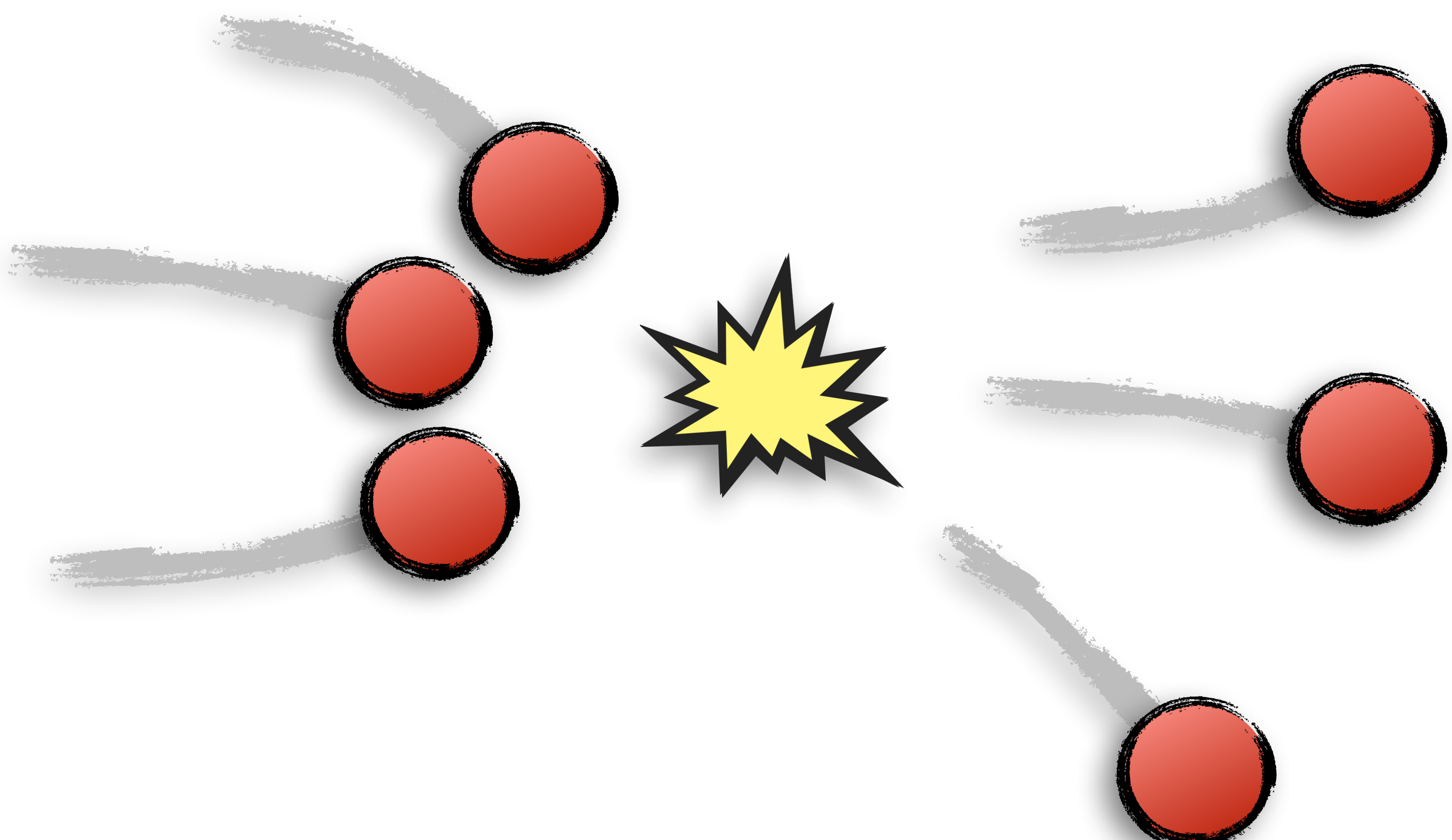
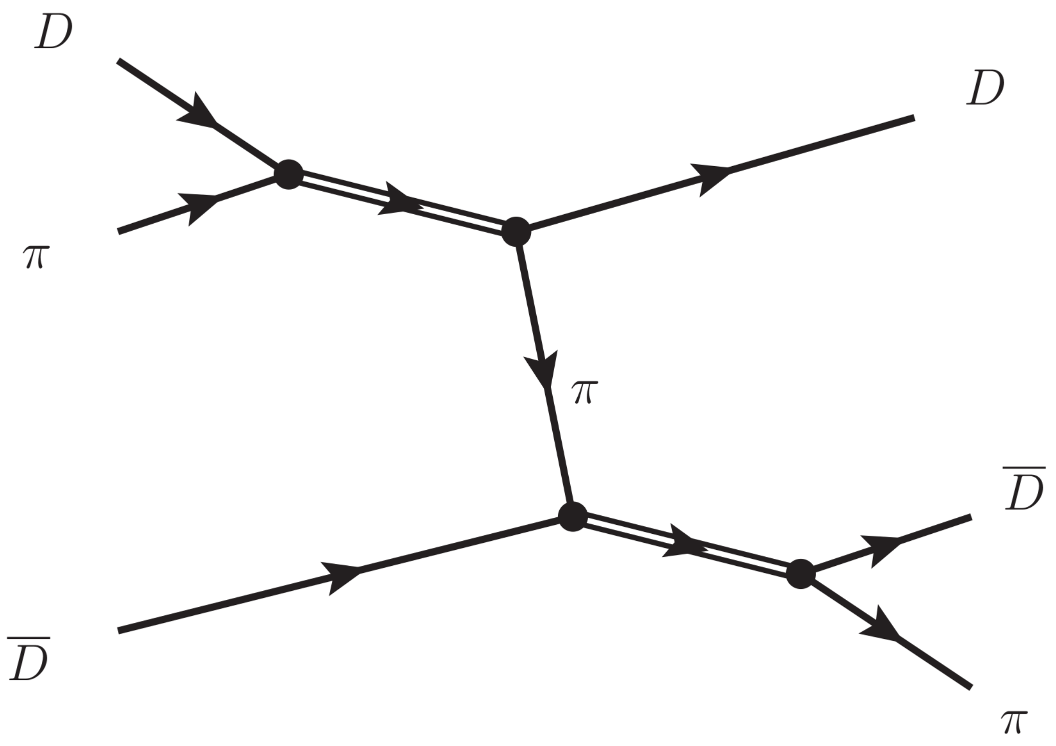
16th International Workshop on Meson Physics
Wednesday 19th, May 2021



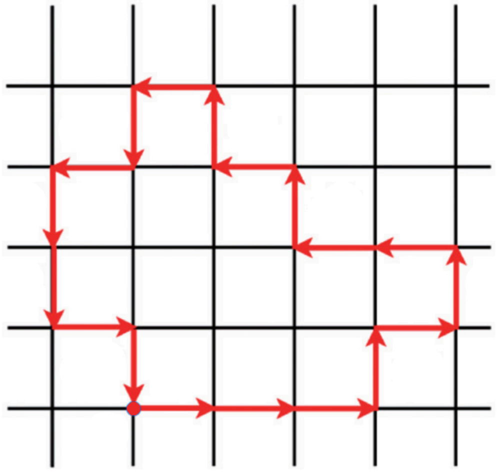
Indiana University
Bloomington

Three-body scattering amplitudes?

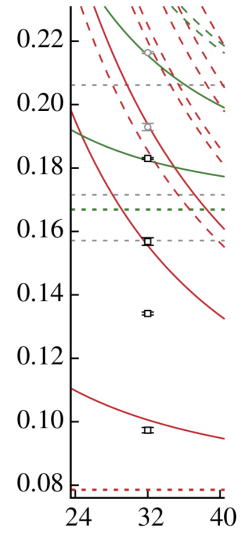
- Exotic resonances decay to three-particle final states
  - X(3872), N(1440), a_1(1420), ...
- Example: interpretations of X(3872)
  - molecule
  - charmonium-molecule hybrid
  - diquark-antidiquark
  - kinematical effect
- Goals
  - Phenomenology
  - Lattice QCD



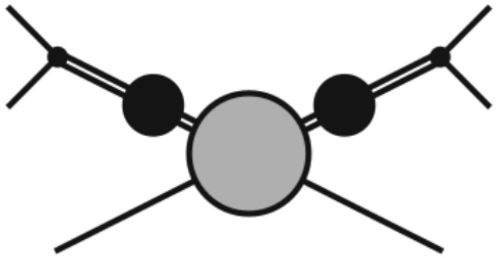
Three-particles interactions from QCD



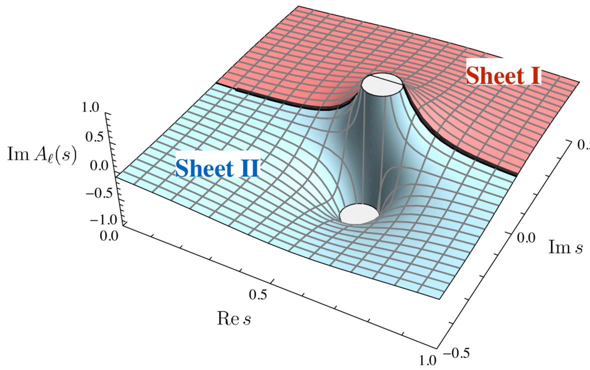
Lattice QCD



FV Spectrum



Amplitudes



Particles Properties

- Finite volume spectrum through the Quantization Condition gives a three-body K-matrix,
- $3 \to 3$ amplitudes obtained from the K-matrix + knowledge of two-body subprocesses, through the set of integral equations,
- Final amplitudes analytically continued to the unphysical Riemann sheets,

Formalisms under construction

- Effective Field Theory framework
  - generic EFT,
  - summation of 2PI, 3PI diagrams,

  Hansen, Sharpe, Phys. Rev. D 90, 116003 (2014),
  Hansen, Sharpe, Phys. Rev. D 92, 114509 (2015),
  Hansen, Sharpe, Phys. Rev. D 95, 034501 (2017),
  and many more...

- Unitarity-based framework
  - parametrization based on the S matrix unitarity,

  Mai et al., Eur. Phys. J. A 53, 177 (2017),
  Mai, Döring, Eur. Phys. J. A 53, 240 (2017),
  Jackura et al., Eur. Phys. J. C 79, no. 1, 56 (2019),
  and many more...

- First results – three pions at I=3

  Blanton et al., Phys. Rev. Lett. 124 (2020) 3, 032001
  Hansen et al., Phys. Rev. Lett. 126 (2021) 012001

- First results – three pions at I=3

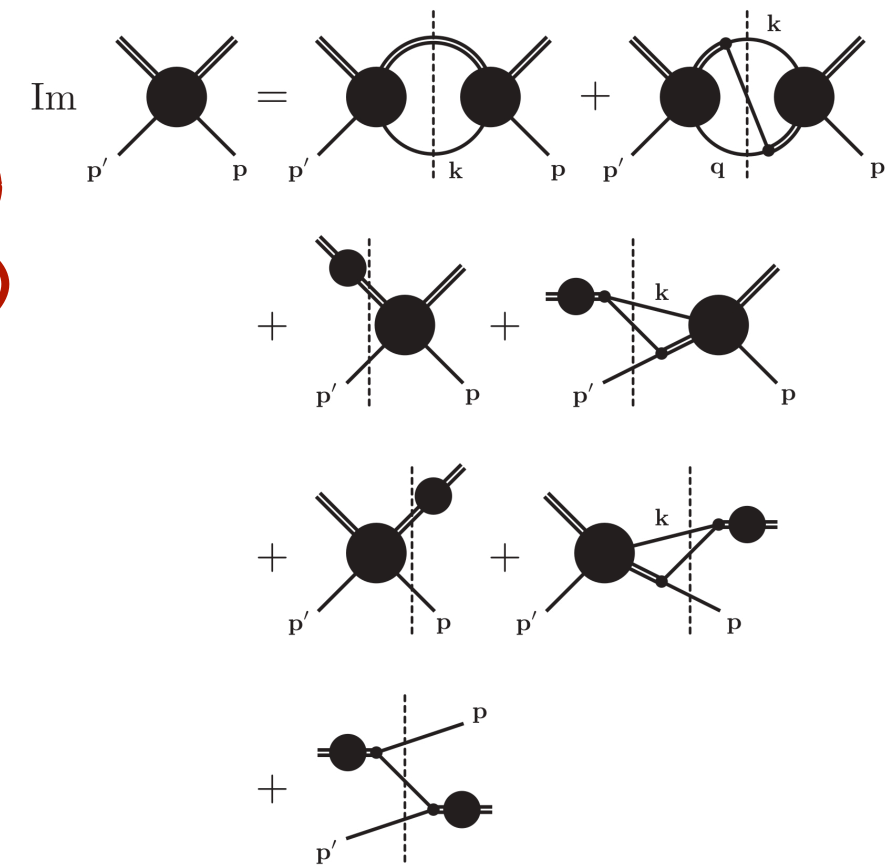
  Mai, Döring, Phys. Rev. Lett. 122, 062503 (2019)
  Culver et al., Phys. Rev. D 101 (2020) 11, 114507

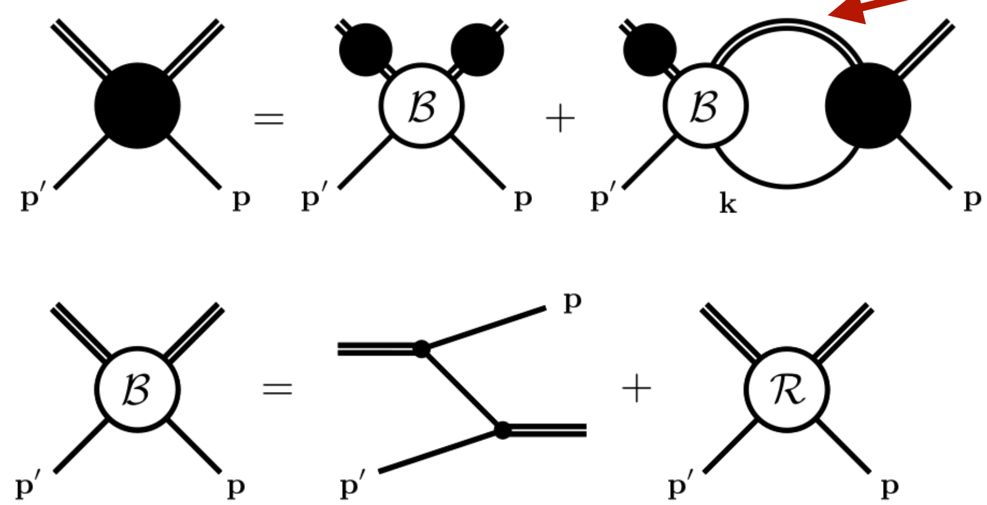
They are equivalent, both in their IV and FV versions:

Jackura et al. Phys. Rev. D 100, 034508 (2019) and Blanton, Sharpe, Phys.Rev.D 102 (2020) 5, 054515

The B-matrix approach

- Starting with the S matrix unitarity
- Physical degrees of freedom (domain of integration)
- Simple parametrization with clear interpretation





One Particle Exchange

Short Range Interactions

$$\mathcal{A} = \mathcal{F}\, B\, \mathcal{F} + \mathcal{F} \int B\, \mathcal{A}$$

Three-body amplitude

$$\mathcal{A}_{\ell m_\ell;\ell' m_\ell}(\sigma', s, \sigma)$$

- pair-spectator,
- partial waves,
- symmetrization,

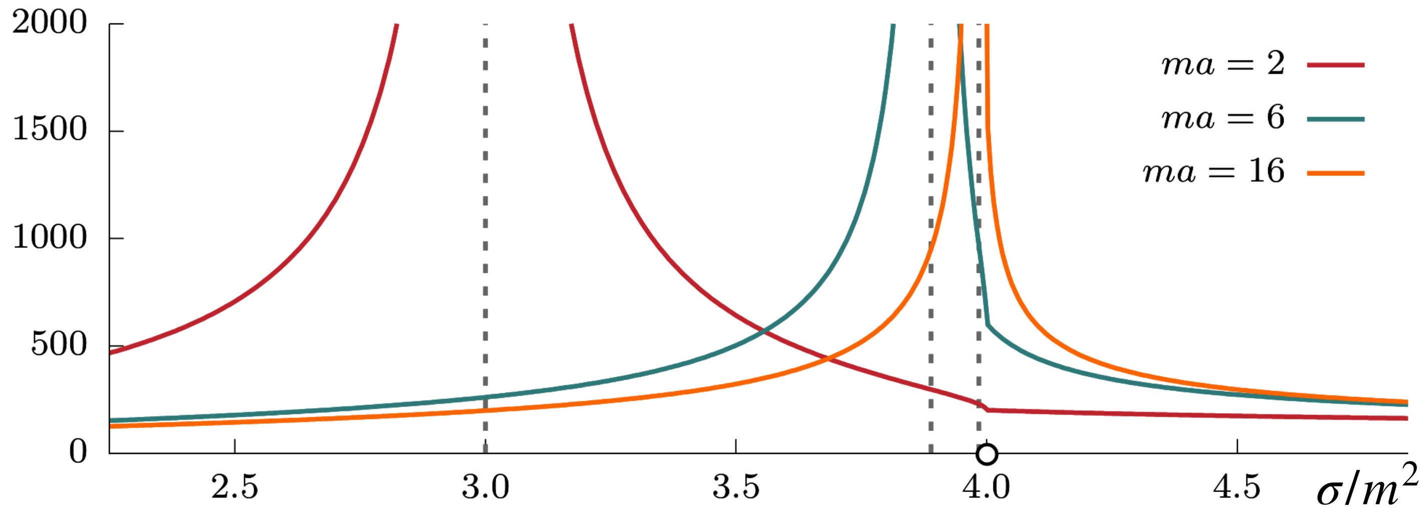
Bound-state–particle scattering

- Model study: formation of the three-body bound states
- Properties of the integral equations and analytic continuation
- Simple model: two-particle bound state + particle, S wave
  - Analytic properties of the B-matrix formalism

    Phys.Rev.D 103 (2021) 1, 014009 (with A. Szczepaniak)

  - Numerical solution of the three-body EFT equations

    arXiv:2010.09820 (with A. Jackura, R. Briceño, M. H. E Islam, C. McCarty)

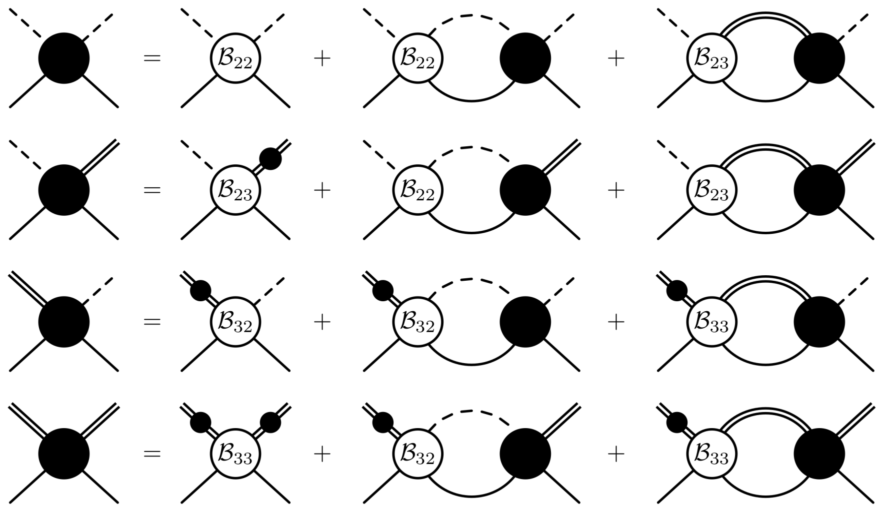


$$\mathcal{F}^{-1} \sim \frac{1}{a} + i\rho$$

$$\lim_{\sigma,\sigma'\to\sigma_b} \mathcal{A}_3 = \frac{g}{\sigma' - \sigma_b}\,\mathcal{A}_2\,\frac{g}{\sigma - \sigma_b}$$

Generalization of the B-matrix equations

- One can not use the LSZ reduction – bound-state energies outside of physical region
- Multi-hadron scattering requires generalization to all channels



$$\mathcal{A}_{22} = \mathcal{B}_{22} + \int_{\hat{\boldsymbol{k}}} \mathcal{B}_{22} i\rho_2 \mathcal{A}_{22} + \int_{\boldsymbol{q}} \mathcal{B}_{23,\boldsymbol{q}}\, \mathcal{A}_{32,\boldsymbol{q}},$$

$$\mathcal{A}_{23,\boldsymbol{p}} = \mathcal{B}_{23,\boldsymbol{p}}\, \mathcal{F}_{\boldsymbol{p}} + \int_{\hat{\boldsymbol{k}}} \mathcal{B}_{22} i\rho_2 \mathcal{A}_{23,\boldsymbol{p}} + \int_{\boldsymbol{q}} \mathcal{B}_{23,\boldsymbol{q}}\, \mathcal{A}_{33,\boldsymbol{qp}},$$

$$\mathcal{A}_{32,\boldsymbol{p}'} = \mathcal{F}_{\boldsymbol{p}'}\, \mathcal{B}_{32,\boldsymbol{p}'} + \int_{\hat{\boldsymbol{k}}} \mathcal{F}_{\boldsymbol{p}'} \mathcal{B}_{32,\boldsymbol{p}'} i\rho_2 \mathcal{A}_{22} + \int_{\boldsymbol{q}} \mathcal{F}_{\boldsymbol{p}'}\, \mathcal{B}_{33,\boldsymbol{p}'\boldsymbol{q}}\, \mathcal{A}_{32,\boldsymbol{q}},$$

$$\mathcal{A}_{33,\boldsymbol{p}'\boldsymbol{p}} = \mathcal{F}_{\boldsymbol{p}'}\, \mathcal{B}_{33,\boldsymbol{p}'\boldsymbol{p}}\, \mathcal{F}_{\boldsymbol{p}} + \int_{\hat{\boldsymbol{k}}} \mathcal{F}_{\boldsymbol{p}'}\, \mathcal{B}_{32,\boldsymbol{p}'} i\rho_2 \mathcal{A}_{23,\boldsymbol{p}} + \int_{\boldsymbol{q}} \mathcal{F}_{\boldsymbol{p}'}\, \mathcal{B}_{33,\boldsymbol{p}'\boldsymbol{q}}\, \mathcal{A}_{33,\boldsymbol{qp}} \,.$$

- Satisfies unitarity above the three-particle threshold,
- **Approximation:** all multi-particle interactions are constant and real (couplings g_{ij}),

$$a_{33}(s) = \frac{g_{33}}{1 - g_{22} i\rho_2 - g_{33}\mathcal{I}(s)}, \qquad a_{22}(s) = \frac{g_{22}}{1 - g_{22} i\rho_2 - g_{33}\mathcal{I}(s)}.$$

Analyticity of the B-matrix equations

- Solutions do not satisfy unitarity below the three-body threshold
- Spurious singularities start arbitrarily close to the two-body threshold

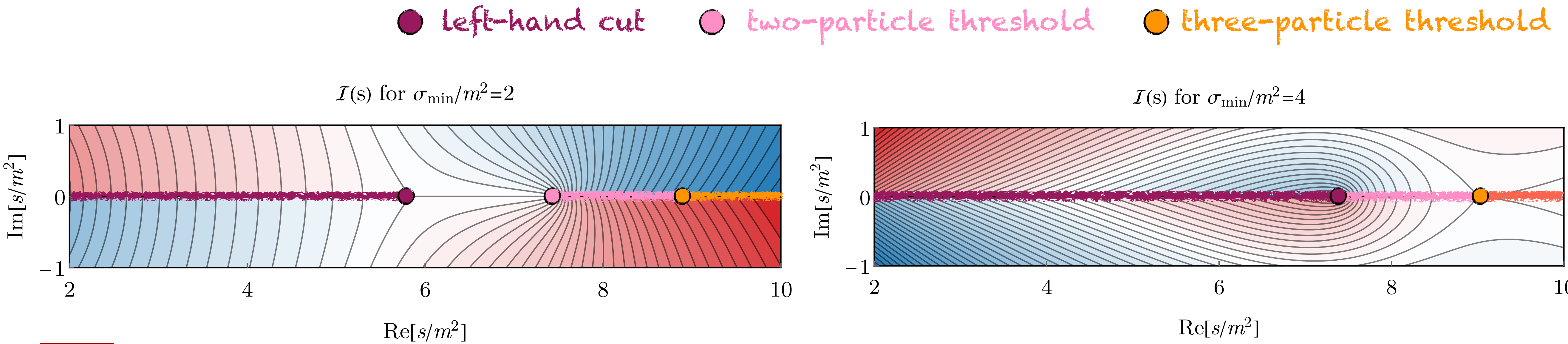
Kernel suffering from non-physical left-hand cuts:

$$\mathcal{I}(s) = \int_{\sigma_{\min}}^{(\sqrt{s}-m)^2} \frac{d\sigma_{\boldsymbol{q}}}{2\pi}\, \tau(s,\sigma_{\boldsymbol{q}})\, \mathcal{F}(\sigma_{\boldsymbol{q}})$$

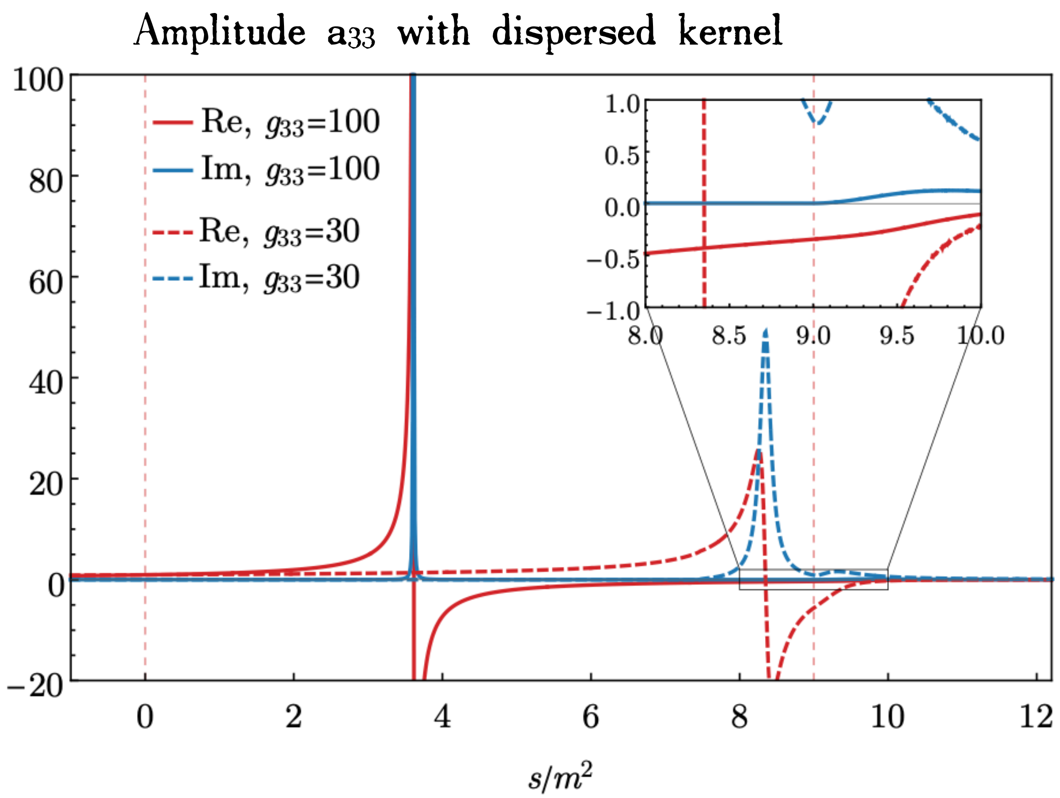
- Dispersion procedure ensures analyticity

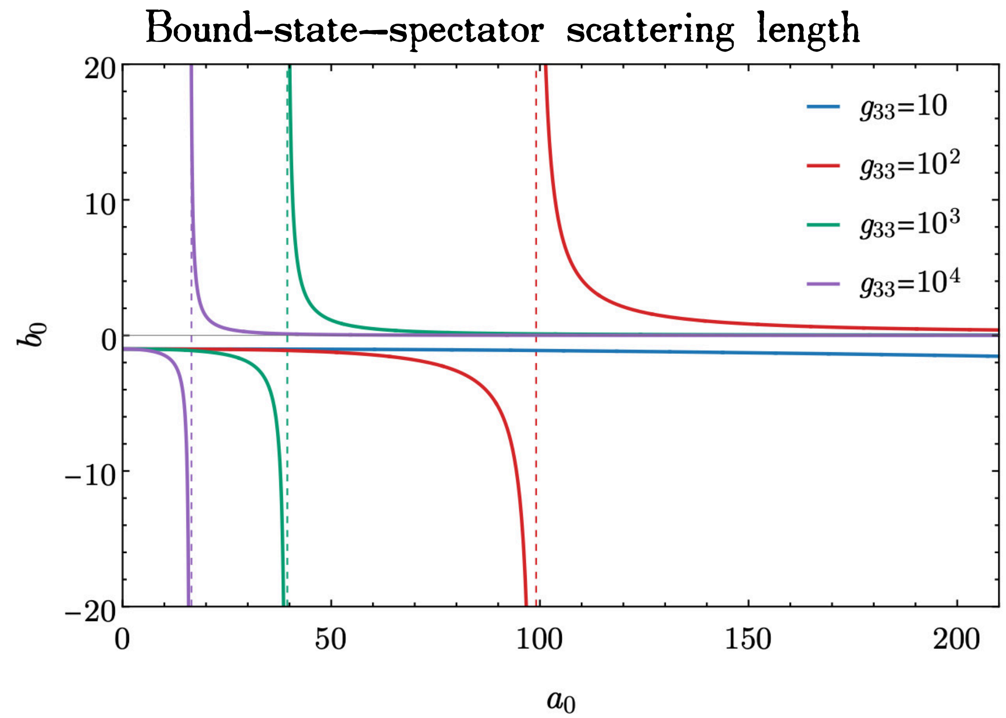
Kernel free from those problems:

$$\mathcal{I}_d(s) = \frac{(s-s_s)^2}{\pi} \int_{(3m)^2}^{\infty} ds' \, \frac{\mathrm{Im}\, \mathcal{I}(s')}{(s'-s-i\epsilon)(s'-s_s-i\epsilon)^2}$$



Results



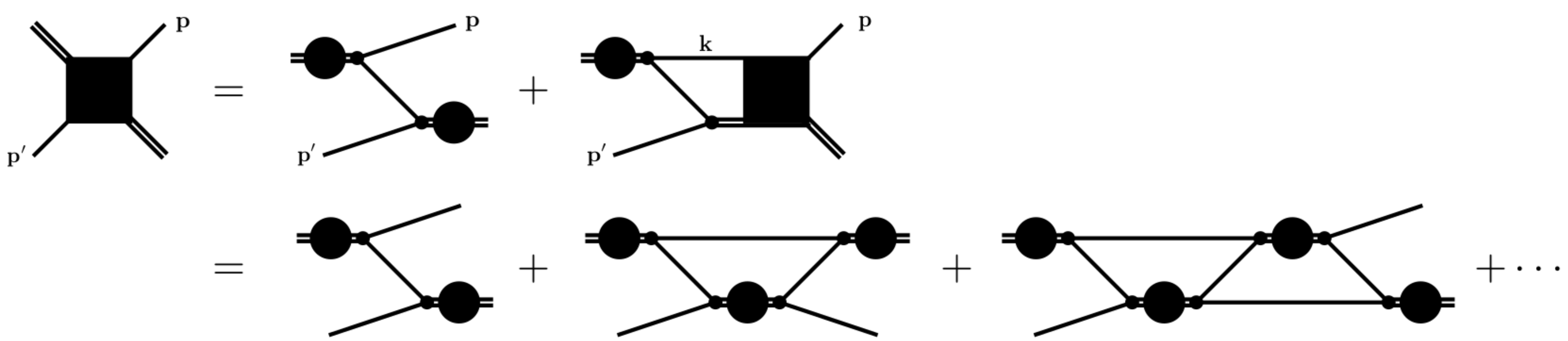


$$b_0 = -\frac{g_{22}}{\left(1 - g_{33}\,\mathcal{I}_d(s_{\text{th},2})\right)}$$

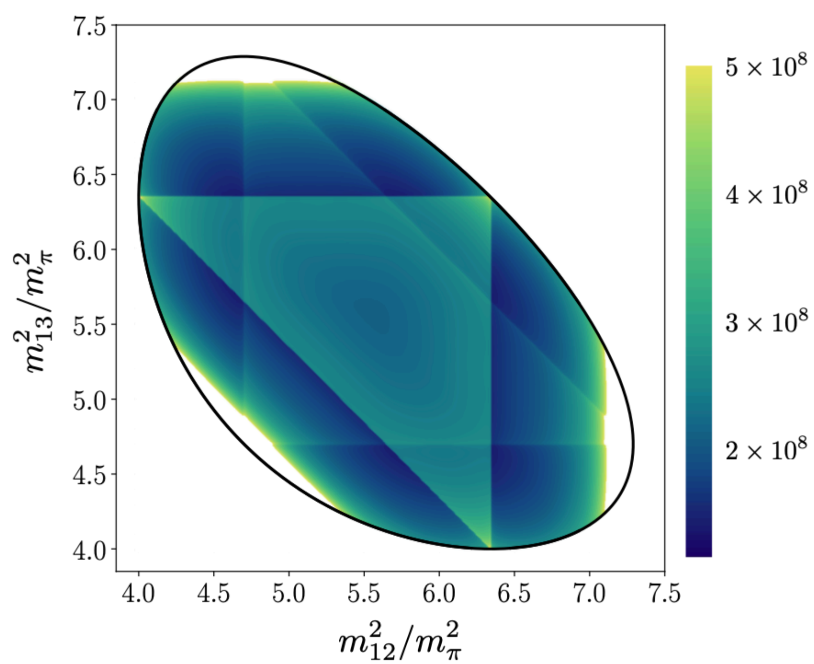
- Spurious singularities pushed to non-physical Riemann sheets, can study physics,
- General dispersion procedure for the three-body unitarity formalism is needed,

Solving the EFT three-body ladder equation

- Ladder approximation, B = G + (R=0)



- Calculation above the bound-state—particle threshold,



- weakly interacting system in the $\pi^+\pi^+$ and $\pi^+\pi^+\pi^+$

  Hansen et al., Phys. Rev. Lett. 126 (2021), 012001

- decay $a_1(1260) \to \rho^0\pi^- \to \pi^-\pi^+\pi^-$

  Sadasivan et al., Phys.Rev.D 101 (2020) 9, 094018

Numerical procedure

- Discretization of the integral equation $\longrightarrow$ N linear equations (Matrix equation)
- Regulation of the bound-state pole via ϵ-prescription

$$\mathcal{A}_2(s) = \lim_{\epsilon\to 0^+} \lim_{N\to\infty} \mathcal{A}_2(s;N,\epsilon)$$

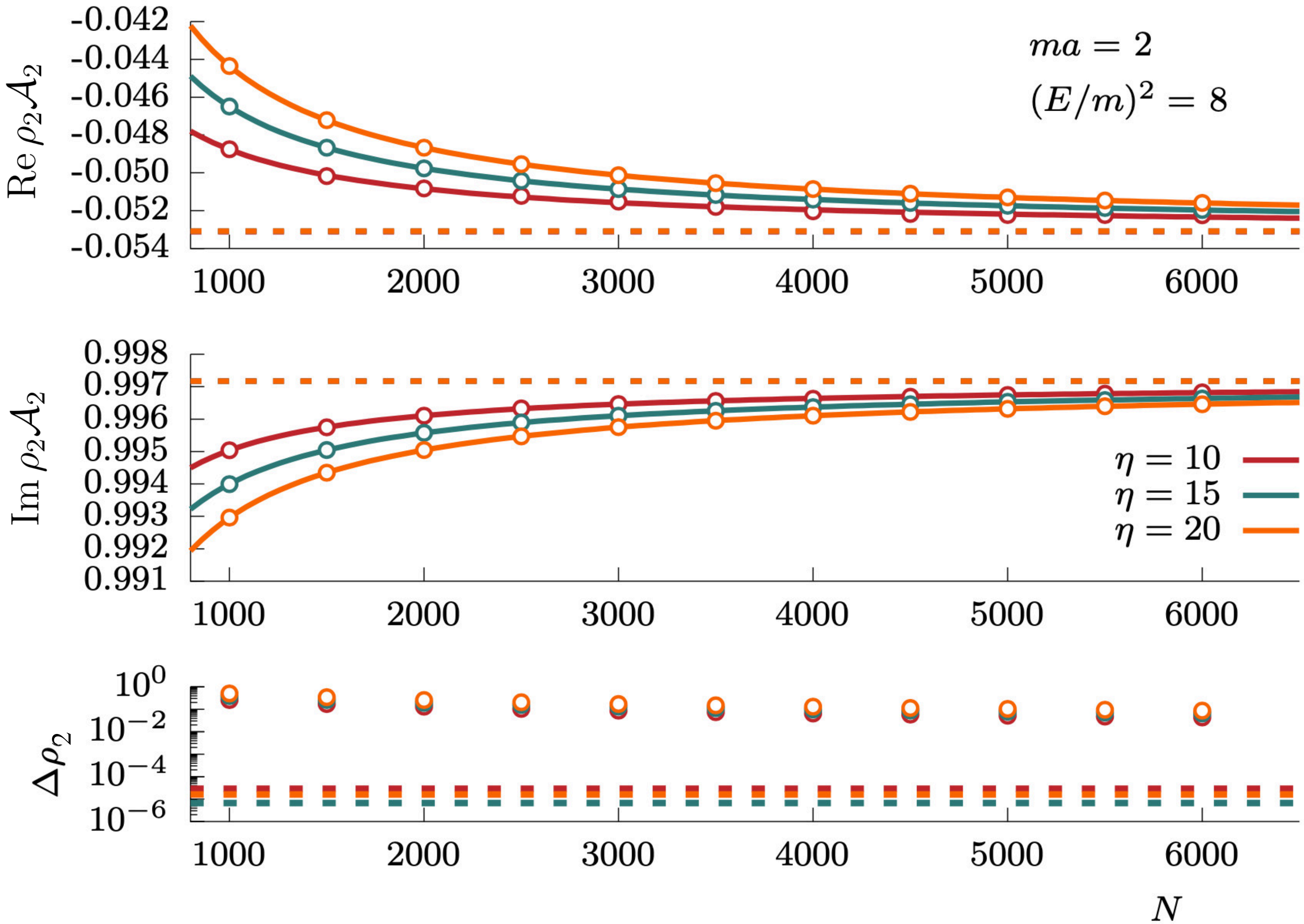
- Systematics:
  - **Unitarity test:** $\mathrm{Im}\mathcal{A}_2(s) = \rho_2(s)|\mathcal{A}_2(s)|^2 \longrightarrow \Delta\rho_2 = 100 \times \left|\frac{\mathrm{Im}\,\mathcal{A}_2^{-1}(s;N,\epsilon) + \rho_2(s)}{\rho_2(s)}\right|$
  - **Convergence test:** $\Delta_N\mathcal{A}_2 = 2 \times \left|\frac{\mathcal{A}_2(s;N+1,\epsilon) - \mathcal{A}_2(s;N,\epsilon)}{\mathcal{A}_2(s;N+1,\epsilon) + \mathcal{A}_2(s;N,\epsilon)}\right|$

- Methods:

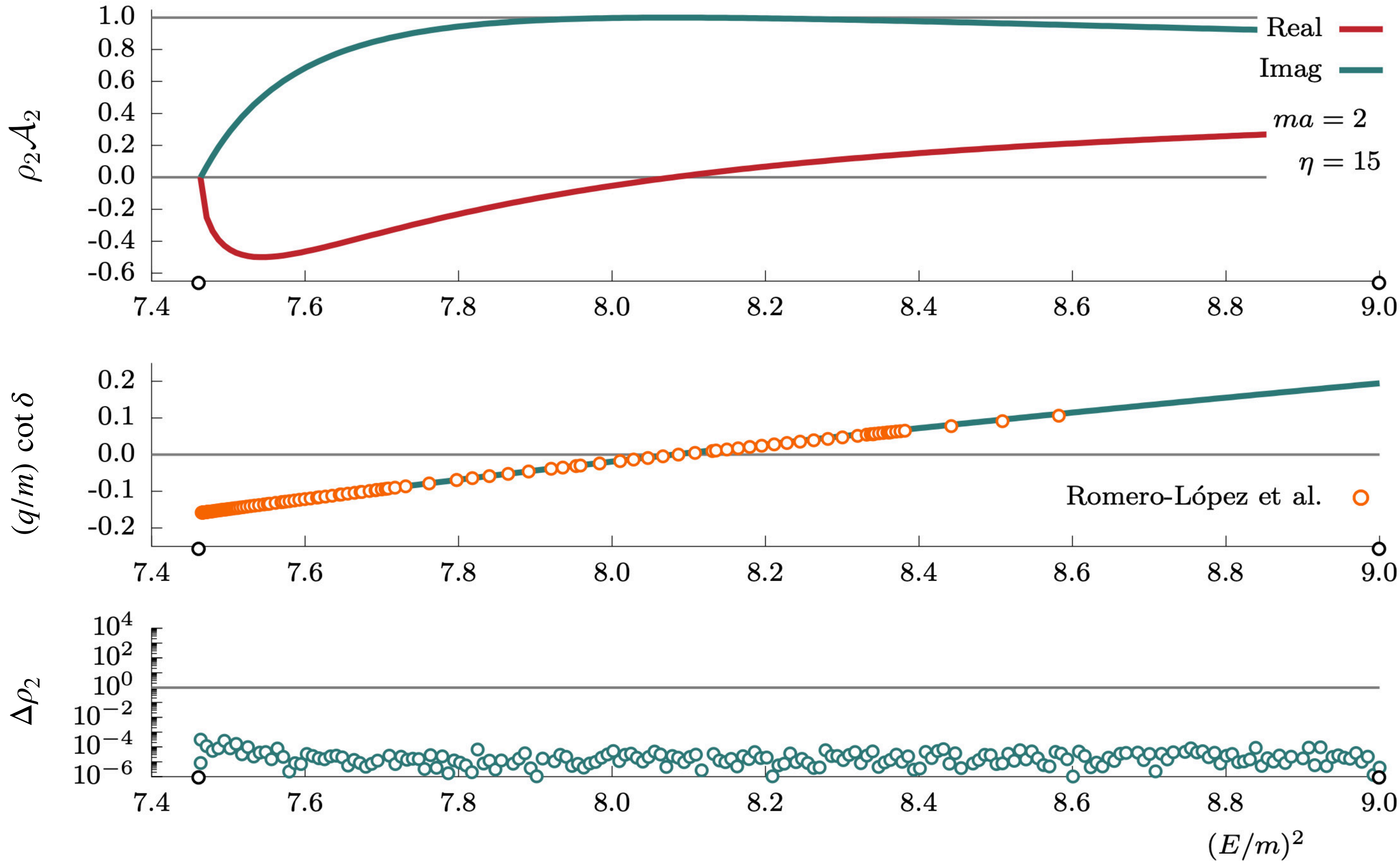
“Brute force” | Explicit pole removal | Spline-based quadratures

Extrapolation in N

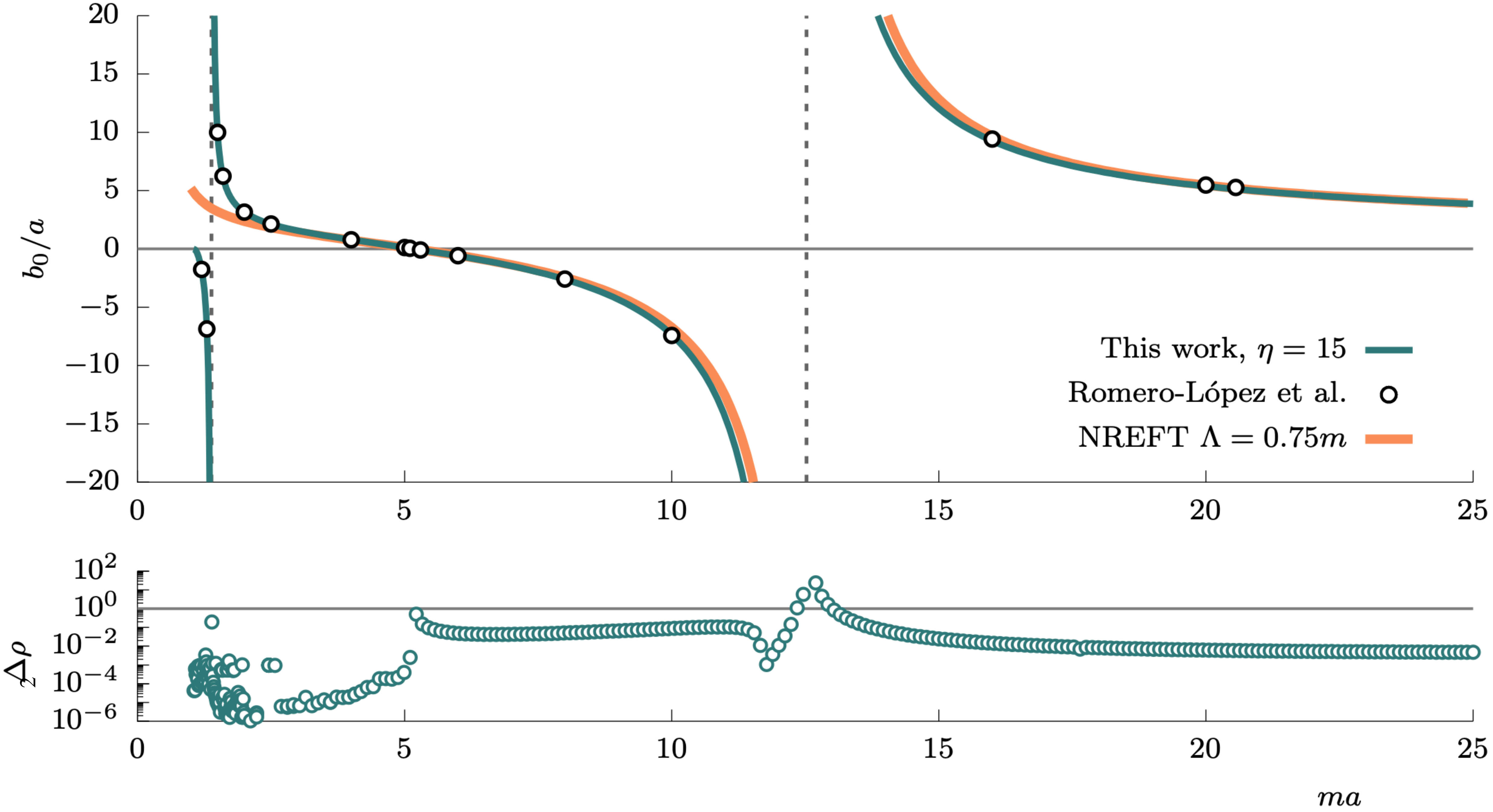
- Expansion in 1/N,
- Epsilon regulator fixed to $\epsilon \propto \eta/N$



Example results, $M^2 = 3m^2$



Example result, three-body bound state



Romero-Lopez et al., JHEP 10 (2019) 007

Bedaque et al., Nucl. Phys. A 646 (1999) 444

Conclusions

- First lattice calculations of the 3-hadron systems
- Systematic procedure for solving the integral equations

Future work

- Continuing below the two-body threshold and to complex energies
- Formulation of the B-matrix satisfying analyticity
- Generalization to arbitrary spins
- Controlling the scheme/ cutoff dependence

BACKUP SLIDES

Example results, $2 \to 3$ amplitude

- We are not limited below the three-body threshold,